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Model theory of a Hilbert space expanded with an unbounded closed selfadjoint operator

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We study a closed unbounded self-adjoint operator Q acting on a Hilbert space H in the framework of *Metric Abstract Elementary Classes* (MAECs). We build a suitable MAEC for such a structure, prove it is $\aleph_0$-categorical and $\aleph_0$-stable up to a system of perturbations. We give an explicit continuous $\mathcal{L}_{\omega_1,\omega}$ axiomatization for the class. We also characterize non-splitting and show it has the same properties as non-forking in superstable first order theories. Finally, we characterize equality, orthogonality and domination of (Galois) types in that MAEC.

1 Introduction

This paper deals with a complex Hilbert space expanded by a unbounded closed self-adjoint operator Q, from the point of view of *Metric Abstract Elementary Classes* (cf. [17]).

Previous work related to this paper can be classified into two types: work of the first type deals with the model theory of Hilbert spaces expanded with some operators in the framework of continuous logic; work of the second type is about the development of a notion of *Abstract Elementary Class* similar to Shelah's (cf. [24]), but suitable for analytic structures along with its further analysis.

Previous work of the first type goes back to Iovino's doctoral thesis (cf. [19]), where he and his advisor Henson noticed that the structure $(H, 0, +, \langle\ |\ \rangle, A)$, where A is a bounded operator, is stable. In [11], Berenstein and Buechler gave a geometric characterization of forking in those structures, when the operator is unitary, after adding to it the projections determined by the Spectral Decomposition Theorem. Ben Ya'acov, Usvyatsov and Zadka (cf. [9]) worked on the first order continuous logic theory of a Hilbert space with a generic automorphism, and characterized the generic automorphisms on a Hilbert space as those whose spectrum is the unit circle. The author and Berenstein (cf. [5]) studied the theory of the structure $(H, +, 0, \langle\ |\ \rangle, U)$ where U is a unitary operator in the case when the spectrum is countable. The author and Ben Ya'acov (cf. [4]), studied the case of a Hilbert space expanded by a normal operator N. Finally, in a recently published paper, the author has dealt with non-degenerate representations of an unital (non-commutative) C^*-algebra (cf. [3]).

Concerning work of the second type: in the 1980s, Shelah defined the notion of *Abstract Elementary Classes* (AEC) as a generalization of the notion of elementary classes, which is a class of models of a first order theory [24]. Shelah's paper generated a big trend in model theory towards the study of these classes. In order to deal with the case of analytic structures, Hyttinen and Hirvonen defined *metric abstract elementary classes* in [17] as a generalization of Shelah's AECs to classes of metric structures (MAECs). After this, Villaveces and Zambrano studied notions of independence and superstability for *metric abstract elementary clases* (MAECs) [25, 26].

The main results in this paper are the following:

1. We build a MAEC associated with the structure (H, Γ_Q) which is denoted by $\mathcal{K}_{(H,\Gamma_Q)}$, where Γ_Q stands for the distance to the graph of the operator Q.
2. We characterize (Galois) types of vectors in some structure in $\mathcal{K}_{(H,\Gamma_Q)}$, in terms of spectral measures.
3. We show that $\mathcal{K}_{(H,\Gamma_Q)}$ is $\aleph_0$-categorical and $\aleph_0$-stable up to a system of perturbations.

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4. We characterize continuous first order elementary equivalence of structures of the type (H, Γ_Q). Incidentally, we give an alternative proof of a famous consequence of Weyl-von Neumann-Berg Theorem.
5. We give a continuous $\mathcal{L}_{\omega_1,\omega}$ axiomatization of the class $\mathcal{K}_{(H,\Gamma_Q)}$.
6. We characterize non-splitting in $\mathcal{K}_{(H,\Gamma_Q)}$ and we show that it has the same properties as non-forking for superstable first order theories.

This paper is divided as follows: In § 2, we give an introduction to Spectral Theory of unbounded closed selfadjoint operators. In § 3, we define a *metric abstract elementary class* associated with (H, Γ_Q) (denoted by $\mathcal{K}_{(H,\Gamma_Q)}$). In § 4, we give a characterization of definable and algebraic closures. In § 5, we define a system of perturbations for $\mathcal{K}_{(H,\Gamma_Q)}$, and show that the class is $\aleph_0$-categorical up to the (previously defined) system of perturbations. In § 6, we give a characterization of first order elementary equivalence and give a continuous $\mathcal{L}_{\omega_1,\omega}$ axiomatization of the class $\mathcal{K}_{(H,\Gamma_Q)}$. As a by product of this, we get an alternate proof of an important cosequence of Weyl-von Neuman-Berg that states that two operators are approximately unitarily equivalent if and only if their essential and discrete spectra coincide and the dimensions of the eigenspaces of their eigenvalues are the same. This fact is proved by using $\aleph_0$-categoricity up to the system of perturbations proved in § 5. In § 7, we prove superstability of the MAEC $\mathcal{K}_{(H,\Gamma_Q)}$. In § 8, we define spectral independence in $\mathcal{K}_{(H,\Gamma_Q)}$ and we show that it is equivalent to non-splitting and has the same properties as non-forking for superstable first order theories. Finally in § 9, we characterize domination, orthogonality of types in terms of absolute continuity and mutual singularity between spectral measures.

2 Preliminaries: Spectral theory of a closed unbounded self-adjoint operator

The following is a small review of spectral theory of a closed unbounded self-adjoint operator; the main sources for this section are [15, 22].

Definition 2.1 Let H be a complex Hilbert space. A *linear operator on* H is a function $S : D(S) \to H$ such that $D(S)$ is a dense vector subspace of H and for all $v, w \in S$ and $\alpha, \beta \in \mathcal{C}$, $S(\alpha v + \beta w) = \alpha S v + \beta S w$.

Definition 2.2 Let S be a linear operator on H. The operator S is called *bounded* if the set $\{\|Su\| : v \in D(S), \|v\| = 1\}$ is bounded in $\mathbb{C}$. If S is not bounded, it is called *unbounded*.

Definition 2.3 If S is bounded we define the *norm* of S by:

$$\|S\| = \sup_{u \in D(S), \|u\|=1} \|Su\|$$

For H a Hilbert space, we denote by $\mathrm{B}(H)$ the algebra of all bounded linear operators on H such that $D(S) = H$.

Definition 2.4 Let R and S be linear operators on H and let $\alpha \in \mathcal{C}$. Then the linear operators $R + S$, αS and S^{-1} are defined as follows:

1. If $D(R) \cap D(S)$ is dense in H, $D(R + S) := D(R) \cap D(S)$ and $(R + S)v := Rv + Sv$ for $v \in D(R + S)$.
2. $D(RS) := \{v \in H \mid v \in D(S) \text{ and } Sv \in D(R)\}$, $(RS)v := R(Sv)$ if $D(RS)$ is dense and $v \in D(RS)$.
3. If $\alpha = 0$, then $\alpha T \equiv 0$ in H. If $\alpha \neq 0$, $D(\alpha S) := D(S)$ and $(\alpha S)v := \alpha S v$ if $v \in D(S)$.
4. If S is one-to-one and $SD(S)$ is dense in H, $D(S^{-1}) := SD(S)$ and $S^{-1}v := w$ if $w \in D(S)$ and $Sw = v$.

Definition 2.5 Let $S : D(S) \to H$ be a linear operator on H. The operator S is called *closed* if the set $\{(v, Sv) \mid v \in D(S)\}$ is closed in $H \times H$. The operator S is called *closable* if the closure of the set $\{(v, Sv) \mid v \in D(S)\}$ is the graph of some operator which is called the *closure* of S and is denoted by $\bar{S}$.

Definition 2.6 Let S be an operator (either bounded or unbounded), and λ a complex number. Then,

1. λ is called a *eigenvalue* of S if the operator $S - \lambda I$ is not one to one. The *point spectrum* of S, denoted by $\sigma_{\mathrm{p}}(S)$, is the set of all the eigenvalues of S.

2. λ is called a *continuous spectral value* if the operator $S - \lambda I$ is one to one, the operator $(S - \lambda I)^{-1}$ is densely defined but is unbounded. The *continuous spectrum* of S, denoted by $\sigma_c(S)$, is the set of all the continuous spectral values of S.
3. λ is called a *residual spectral value* if $(S - \lambda I)H$ is not dense in H. The *residual spectrum* of S, denoted by $\sigma_r(S)$, is the set of all the residual spectral values of S.
4. The *spectrum* of S, denoted by $\sigma(S)$, is the union of $\sigma_p(S)$, $\sigma_c(S)$ and $\sigma_r(S)$.
5. The *resolvent set* of S, denoted by $\varrho(S)$, is the set $\mathcal{C}\backslash\sigma(S)$.
6. If $\lambda \in \varrho(S)$, the *resolvent operator of S at λ* is the operator $(S - \lambda I)^{-1}$, and is denoted by $R_\lambda(S)$.

Definition 2.7 Given linear operators $S : D(S) \to H$ and $S' : D(S') \to H$ on H, S' is said to be an *adjoint operator* of S if for every $v \in D(S) w \in D(S')$, $\langle Sv \mid w\rangle = \langle v|S'w\rangle$.

Definition 2.8 Given a linear operator $S : D(S) \to H$ and $S' : D(S') \to H$ on H, then S' is said to be the *adjoint operator* of S, denoted S^*, if S' is maximal adjoint to S, i.e., if S'' is and adjoint operator of S and $S' \subseteq S''$ then $S' = S''$.

Definition 2.9 A linear operator Q on H is called *symmetric* if $Q \subseteq Q^*$. If $Q = Q^*$, Q is called *self-adjoint.*

Fact 2.10 ([15, Lemma XII.2.2]) The spectrum of a self-adjoint operator Q is real and for $\lambda \in \varrho(Q)$, the resolvent $R_l(Q)$ is a normal operator with $R_\lambda(Q)^* = R_{\hat{\lambda}}(Q)$ and $\|R_\lambda(Q)\| \leq |\mathrm{Im}(\lambda)|$.

Fact 2.11 ([22, Theorem VIII.1]) Let Q be an operator on H. Then,

1. the operator Q^* is closed;
2. the operator Q is closable if and only if $D(Q^*)$ is dense in H in which case $\bar{Q} = Q^{**}$;
3. if Q is closable then $(\bar{Q})^* = Q^*$.

Fact 2.12 ([22, Theorem VIII.3]) Let Q be a symmetric operator on H. Then the following statements are equivalent:

1. The operator Q is self-adjoint.
2. The operator Q is closed and $\mathrm{Ker}(Q^* \pm iI) = \{0\}$.
3. $\mathrm{Ran}(Q \pm iI) = H$.

Definition 2.13 A symmetric operator S is called *essentially self-adjoint* if its closure $\bar{S}$ is self-adjoint.

Fact 2.14 ([22, Corollary of Theorem VIII.3]) Let Q be a symmetric operator on H. Then the following statements are equivalent:

1. Q is essentially self-adjoint.
2. $\mathrm{Ker}(Q^* \pm iI) = \{0\}$.
3. $\mathrm{Ran}(Q \pm iI)$ is dense.

Fact 2.15 ([20, Theorem 9.1-2]) Let $Q : H \to H$ be a closed self-adjoint operator on H. Then a number $\lambda \in \mathbb{R}$ belongs to $\sigma(Q)$ if and only if there exists $c > 0$ such that for every $v \in D(Q)$, $\|(Q - \lambda I)v\| \geq c\|v\|$.

Fact 2.15 was originally stated for bounded operators, but its generalization to closed unbounded self-adjoint operators is straightforward and left to the reader. Recall that $\sigma(Q) \subseteq \mathbb{R}$ by Fact 2.10.

Theorem 2.16 ([22, Spectral Theorem Multiplication Form, Theorem VIII.4]) *Let Q be self-adjoint on a Hilbert space H with domain $D(Q)$. Then there are a measure space (X, μ), with μ finite, an unitary operator $U : H \to L^2(X, \mu)$, and a real function f on X which is finite a.e. so that,*

1. *$v \in D(Q)$ if and only if $f(\cdot)(Uv)(\cdot) \in L^2(X, \mu)$.*
2. *If $g \in U(D(Q))$, then $(UQU^{-1}g)(x) = f(x)g(x)$ for $x \in X$.*

Definition 2.17 A self-adjoint operator Q different from the zero operator is called *positive* and we write $Q \geq 0$, if $\langle Qv|v\rangle \geq 0$ for all $v \in \mathcal{H}$.

Theorem 2.18 (**Functional Calculus Form of the Spectral Theorem** [22, Theorem VIII.5]) *Let Q be a closed unbounded self-adjoint operator on H. Then there is a unique map π from the bounded Borel functions on $\mathbb{R}$ into* $\mathrm{B}(H)$ *such that*

1. *π is an algebraic *-homomorphism.*
2. *π is norm continuous, that is, $\|\pi(h)\|_{\mathrm{B}(H)} \leq \|h\|_\infty$.*
3. *Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of bounded Borel functions with $h_n(x) \to x$ for each x and $|h_n(x)| \leq |x|$ for all x and n. Then for any $v \in D(Q)$, $\lim_{n\to\infty} \pi(h_n)v = Qv$.*
4. *Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of bounded Borel functions. If $h_n \to h$ pointwise and if the sequence $\|h_n\|_\infty$ is bounded, then $\pi(h_n) \to \pi(h)$ strongly.*
5. *If $v \in H$ is such that $Qv = v$, then $\pi(h)v = h()v$.*
6. *If $h \geq 0$, then $\pi(h) \geq 0$*

Definition 2.19 Let Ω be a Borel measurable subset of $\mathbb{R}$. By E_Ω we denote the bounded operator $\pi(\chi_\Omega)$ according to Theorem 2.18.

Fact 2.20 ([22, Remark after Theorem VIII.5]) The previously defined projections satisfy the following properties:

1. For every Borel measurable $\Omega \subset \mathbb{R}$, $E_\Omega^2 = E_\Omega$ and $E_\Omega^* = E_\Omega$.
2. $E_\varnothing = 0$ and $E_{(-\infty,\infty)} = I$
3. If $\Omega = \cup_{n=1}^\infty \Omega_n$ with $\Omega_n \cap \Omega_m = \varnothing$ if $n \neq m$, then $\sum_{n=1}^\infty E_{\Omega_n}$ converges to E_Ω in the strong topology.
4. $E_{\Omega_1} E_{\Omega_2} = E_{\Omega_1\cap\Omega_2}$ (and therefore E_{Ω_1} commutes with E_{Ω_2}) for all Borel measurable $\Omega_1, \Omega_2 \subseteq \mathbb{R}$.

Definition 2.21 The family $\{E_\Omega \mid \Omega \subseteq \mathbb{R}$ is Borel measurable$\}$ described in Fact 2.20 is called the *spectral projection valued measure* (s.p.v.m.) generated by Q.

Fact 2.22 Let X be a locally compact Hausdorff space in which every open set is a countable union of compact sets. Let λ any positive Borel measure on X such that $\lambda(K) < \infty$ for any compact set K. Then λ is regular.

Fact 2.23 ([22, Remark before Theorem VIII.6]) Let $v \in \mathcal{H}$. Then the set function such that for every Borel set $\Omega \subset \mathbb{R}$ assigns the value $\langle E_\Omega v|v\rangle$ is a Borel measure. In the case when $\Omega = (-\infty, \lambda)$, this measure is denoted $\langle E_\lambda v|v\rangle$.

Fact 2.24 (**Integral Decomposition Form of the Spectral Theorem** [22, Theorem VIII.6]) Let Q be a closed unbounded self-adjoint operator on H and let h be a (possibly unbounded) Borel measurable function on $\mathbb{R}$. Then the (possibly unbounded) operator $h(Q)$ defined as the only operator such that

$$\langle h(Q)v\ |v\rangle := \int_{-\infty}^{\infty} h(l)d\langle E_\lambda v \mid v\rangle,$$

whenever $v \in D(h(Q))$, with

$$D(h(Q)) := \{v \in h \mid \int_{-\infty}^{\infty} |h(l)|^2 d\langle E_\lambda v \mid v\rangle < \infty\},$$

is such that $h(Q)$ satisfies properties 1–4 of Theorem 2.18 and if h is a bounded Borel measurable function on $\mathbb{R}$, then $h(Q)$ is exactly the operator $\pi(h)$ described in Theorem 2.18.

Definition 2.25 The *essential spectrum* of a closed unbounded self adjoint operator Q, denoted by $\sigma_{\mathrm{e}}(Q)$, is the set of complex values λ such that for every bounded operator S on H and every compact operator K on H, we have that $(Q - \lambda I)S \neq I + K$.

Let Q be a closed unbounded self-adjoint operator on H. Then $\sigma_{\mathrm{e}}(Q) \subseteq \sigma(Q)$.

The next theorem is known as *Weyl's Criterion*. It gives a useful tool to identify the essential spectrum:

Theorem 2.26 *Let Q be a closed unbouned self-adjoint operator. Then, for every $\lambda \in \mathbb{R}$, the following conditions are equivalent:*

1. $\lambda \in \sigma_{\mathrm{e}}(Q)$
2. *For every* $\varepsilon > 0$, $\dim(E_{(\lambda-\varepsilon,\lambda+\varepsilon)}H) = \infty$

Proof. "(i) $\Rightarrow$ (ii)": Assume that there is an $\varepsilon > 0$ such that $E_{(\lambda-\varepsilon,\lambda+\varepsilon)}H$ finite dimensional. Let

$$h(x) = \frac{1 - \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(x)}{x - \lambda}.$$

Then h is a bounded Borel measurable function on $\mathbb{R}$. By Fact 2.18 (functional calculus), we have that

$$h(Q)(Q - \lambda I) = (Q - \lambda I)h(Q) = I - \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(Q) = I - E_{(l-\varepsilon,\lambda+e)}H.$$

Since $E_{(\lambda-\varepsilon,\lambda+\varepsilon)}(Q)$ is finite dimensional, it is compact and $\lambda \notin \sigma_{\mathrm{e}}(Q)$

"(ii) $\Rightarrow$(i)": Suppose that $\lambda \notin \sigma_{\mathrm{e}}(Q)$. Then there are a bounded operator S and a compact operator K such that

$$S(Q - \lambda I) = (Q - \lambda I)S = I + K. \tag{$*$}$$

Suppose that for some $v \in H$, $(Q - \lambda I)v = 0$. Then $(I - K)v = 0$ and, therefore, $Kv = -v$. Since K is compact, this implies that $\mathrm{Ker}(Q - \lambda I)$ is finite dimensional by the hypothesis, for all $\varepsilon > 0$, $\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(Q)$ is infinite dimensional and contains $\mathrm{Ker}(Q - \lambda I)$ which is finite dimensional. So, for every $\varepsilon > 0$ there exists $v_\varepsilon \in \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(Q)$ such that $\|v_\varepsilon\| = 1$ and $d(v_\varepsilon, \mathrm{Ker}(Q - \lambda I)) = 1$ By Theorem 2.24

$$\begin{aligned}\|(Q - \lambda I)v_\varepsilon\|^2 &= \langle (Q - \lambda I)^*(Q - \lambda I)\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(Q)(v_\varepsilon)|v_\varepsilon\rangle \\ &= \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |x - \lambda|^2 d\langle E_x v_\varepsilon \mid v_\varepsilon\rangle \le \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |x - \lambda|^2 dx \le \varepsilon^2 \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} dx \le 2\varepsilon^3\end{aligned}$$

and hence $Qv_\varepsilon - \lambda v_\varepsilon \to 0$ when $\varepsilon \to 0$. From (*) we get:

$$v_\varepsilon + kv_\varepsilon = S(Qv_\varepsilon - \lambda v_\varepsilon) \to 0 \text{ when } \varepsilon \to 0.$$

By compactness of k, there exists a sequence $(v_n) \subseteq \{v_\varepsilon \mid \varepsilon > 0\}$ such that $kv_n \to v$ when $n \to \infty$ for some $v \in H$. It follows that $v_n \to -v$ and, since $\|v_n\| = 1$, we get $\|v\| = 1$. Since $Q(v_n) - \lambda v_n \to 0$ when $n \to \infty$, we get $Qv = \lambda v$, and hence:

$$\|v_n - v\| \ge d(v_n, \mathrm{Ker}(Q - \lambda I)) = 1,$$

which is a contradiction. □

Definition 2.27 Let Q be a closed unbounded self-adjoint operator on H. The *discrete spectrum* of Q is the set:

$$\sigma_{\mathrm{d}}(Q) := \sigma(Q)\backslash\sigma_{\mathrm{e}}(Q)$$

Definition 2.28 Let Q_1 and Q_2 be closed unbounded self-adjoint operators defined on Hilbert spaces H_1 and H_2 respectively. Then (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}) are said to be *spectrally equivalent* ($Q_1 \sim_\sigma Q_2$) if both of the following conditions hold:

1. $\sigma(Q_1) = \sigma(Q_2)$.
2. $\sigma_{\mathrm{e}}(Q_1) = \sigma_{\mathrm{e}}(Q_2)$.
3. $\dim\{x \in H_1 \mid Q_1x = \lambda x\} = \dim\{x \in H_2 \mid Q_2x = \lambda x\}$ for $\lambda \in \sigma(Q_1)\backslash\sigma_{\mathrm{e}}(Q_1)$.

Fact 2.29 ([22, Classical Weyl theorem, Example 3 of § XIII.4]) If Q is a (possibly unbounded) self-adjoint operator and K is a compact operator on H. Then $\sigma_{\mathrm{e}}(Q) = \sigma_{\mathrm{e}}(Q + K)$.

Fact 2.30 ([10, Weyl-Von Neumann-Berg, Corollary 2]) Let Q be a not necessarilly bounded self-adjoint operator on a separable Hilbert space H. Then for every $\varepsilon > 0$ there exists a diagonal operator D and a compact operator K on H such that $\|K\| < \varepsilon$ and $Q = D + K$.

Definition 2.31 Two unbounded closed self-adjoint operators Q_1 and Q_2 on a separable Hilbert spaces H_1 and H_2 are said to be *approximately unitarily equivalent* if there exists a sequence of unitary operators $(U_n)_{n<\omega}$ from H_1 to H_2 such that for every $n \in \mathbb{Z}_+$, $Q_2 - U_n Q_1 U_n^*$ is bounded and for all $\varepsilon > 0$, there is n_ε such that for every $n \geq n_\varepsilon$, $\|Q_2 - U_n Q_1 U_n^*\| < \varepsilon$.

The next theorem is an important consequence of the Weyl-von Neumann-Berg Theorem. In § 6, we shall give a model theoretic proof of it:

Fact 2.32 ([13, II.4.4]) Suppose Q_1 and Q_2 are unbounded closed self-adjoint operators on a separable Hilbert space H. Then Q_1 and Q_2 are approximately unitarily equivalent if and only if $Q_1 \sim_\sigma Q_2$.

Definition 2.33 Let Q be a closed unbounded self-adjoint operator on a Hilbert space H. For $\lambda \in \sigma_{\mathrm{d}}(Q)$, let n_λ be the dimension of the eigenspace corresponding to λ. We define the *discrete part* of H by $H_{\mathrm{d}} := \bigoplus_{\lambda\in\sigma_{\mathrm{d}}(Q)} \mathcal{C}^{n_\lambda}$. In the same way, we define $Q_{\mathrm{d}} := Q \restriction H_{\mathrm{d}}$

Definition 2.34 Let Q be a closed unbounded self-adjoint operator on a Hilbert space H. We define the essential part of H by $H_{\mathrm{e}} := H_{\mathrm{d}}^{\perp}$. In the same way, we define $Q_{\mathrm{e}} := Q \restriction H_{\mathrm{e}}$

Definition 2.35 Given $G \subseteq H$ and $v \in H$, we denote the Hilbert subspace of H generated by the elements $h(Q)v$, where $v \in G$, h is a bounded Borel function on $\mathbb{R}$ and $v \in D(h(Q))$ by H_G. If $v \in H$, we write $H_v := H_{\{v\}}$. We let $Q_G := Q \restriction H_G$ and similarly $Q_v := Q_{\{v\}}$. Furthermore, we write $H_G^{\perp}$ for the orthogonal complement of H_G, P_G for the projection over H_G, and $P_{G^\perp}$ for the projection over $H_G^{\perp}$.

Definition 2.36 Given $G \subseteq H$ and $v \in H$, we denote by $(H_G)_{\mathrm{d}}$ and $(H_G)_{\mathrm{e}}$ the projections of H_G on H_{d} and H_{e} respectively.

Definition 2.37 Let $v \in H$, the *spectral measure defined by* v (denoted by μ_v) is the finite Borel measure that to any Borel set $\Omega \subseteq \mathbb{R}$ assigns the (complex) number,

$$\mu_v(\Omega) := \langle \chi_\Omega(Q)v \mid v\rangle$$

Fact 2.38 ([15, Lemma XII.3.1]) For $v \in H$, the space $H_v \simeq L^2(\mathbb{R}, \mu_v)$.

Fact 2.39 ([15, Lemma XII.3.2]) There is a set $G \subseteq H$ such that $H = \bigoplus_{v\in G} H_v$.

Corollary 2.40 *There is a set $G \subseteq H$ such that $H = H_{\mathrm{d}} \oplus \bigoplus_{v\in G} H_v$.*

3 A metric abstract elementary class defined by $(H; Q)$

In this section we define a *metric abstract elementary class* associated with a closed unbounded self-adjoint operator Q defined on a Hilbert space (cf. Definition 3.4). We shall recall several notions related with metric abstract elementary classes that come from [17].

Definition 3.1 An $\mathcal{L}$-*metric structure* $\mathcal{M}$, for a fixed similarity type $\mathcal{L}$, consists of

(a) a closed metric space (M, d),
(b) a family $(R^{\mathcal{M}})_{R\in\mathcal{L}}$ of continuous functions from M^{n_R} into $\mathbb{R}$, where n_R is the arity of R,
(c) an indexed family $(F^{\mathcal{M}})_{F\in\mathcal{L}}$ of continuous functions on powers of M, and
(d) an indexed family $(c^{\mathcal{M}})_{c\in\mathcal{L}}$ of distinguished elements of M.

We write this structure as

$$\mathcal{M} = (M, d, (R^{\mathcal{M}})_{R\in\mathcal{L}}, (F^{\mathcal{M}})_{F\in\mathcal{L}}, (c^{\mathcal{M}})_{c\in\mathcal{L}}).$$

If $\mathcal{M}$ is a metric structure, $\mathrm{dens}(\mathcal{M})$ denotes the smallest cardinal of a dense subset of M.

Definition 3.2 Let $\mathcal{L} = (0, -, i, +, (I_r)_{r\in\mathbb{Q}}, \|\cdot\|, \Gamma_Q)$. A *Hilbert space operator* structure for $\mathcal{L}$ is a metric structure of only one sort:

$$(H, 0, +, i, (I_r)_{r\in\mathbb{Q}}, \|\cdot\|, \Gamma_Q)$$

where H is a Hilbert space, Q is a closed (unbounded) self-adjoint operator on H, 0 is the zero vector in H, $+ : H \times H \to H$ is the usual sum of vectors in H, $i : H \to H$ is the function that to any vector $v \in H$ assigns the vector iv where $i^2 = -1$, $I_r : H \to H$ is the function that sends every vector $v \in H$ to rv, where $r \in \mathbb{Q}$, $\| \cdot \| : H \to \mathbb{R}$ is the norm function, and $\Gamma_Q : H \times H \to \mathbb{R}$ is the function that to any $v, w \in H$ asigns the number $\Gamma_Q(v, w)$, which is the distance of (v, w) to the graph of Q. Since Q is closed, $\Gamma_Q(v, w) = 0$ if and only if (v, w) belongs to the graph of Q. The structure will be referred to as (H, Γ_Q) and is a metric structure for the similarity type $\mathcal{L}$.

Lemma 3.3 *Let Q_1 and Q_2 be closed unbounded self-adjoint operators defined on Hilbert spaces H_1 and H_2 respectively. An isomorphism $U : (H_1, \Gamma_{Q_1}) \to (H_2, \Gamma_{Q_2})$ is a unitary operator of $U : H_1 \to H_2$ such that $UD(Q_1) = D(Q_2)$ and $UQ_1v = Q_2Uv$ for every $v \in D(Q_1)$.*

P r o o f. "$\Rightarrow$": Suppose U is an isomorphism between (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}). It is clear that U must be a linear operator. Also, we have that for every $u, v \in \mathcal{H}$ we must have that $\langle Uu \mid Uv \rangle = \langle u \mid v \rangle$ by definition of automorphism. Therefore U must be an isometry and, therefore, it must be unitary.

On the other hand, since U is an isomorphism between (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}), for every $(v, w) \in H \times H$ we have that $\Gamma_{Q_1}(v, w) = \Gamma_{Q_2}(Uv, Uw)$. Therefore, $\Gamma_{Q_1}(v, w) = 0$ if and only if $\Gamma_{Q_2}(Uv, Uw) = 0$. So, for every $v \in D(Q_1)$, $UQ_1v = Q_2Uv$.

"$\Leftarrow$": Let $U : H_1 \to H_2$ be an unitary operator such that $UD(Q_1) = D(Q_2)$ and $UQ_1v = Q_2Uv$ for every $v \in D(Q_1)$. It remains to show that for every $(v, w) \in H \times H$, $\Gamma_{Q_1}(v, w) = \Gamma_{Q_2}(Uv, Uw)$. Let $(v, w) \in H \times H$ be any pair of vectors. There exists a sequence of pairs $(v_n, w_n)_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$, $v_n \in D(Q_1)$, $w_n = Q_1v_n$ and $\Gamma_{Q_1}(v, w) = \lim_{n \to \infty} d[(v, w); (v_n, w_n)]$.

By hypothesis, U is an isometry, and maps the graph of Q_1 into the graph of Q_2; so for all $n \in \mathbb{N}$, $Uv_n \in D(Q_2)$ and $Uw_n = Q_2v_n$. We have that

$$\lim_{n \to \infty} d[(Uv, Uw); (Uv_n, Uw_n)] = \lim_{n \to \infty} d[(v, w); (v_n, w_n)] = \Gamma_{Q_1}(v, w).$$

So $\Gamma_{Q_2}(Uv, Uw) \leq \Gamma_{Q_1}(v, w)$. Repeating the argument for U^{-1}, we get $\Gamma_{Q_1}(v, w) \leq \Gamma_{Q_2}(Uv, Uw)$. □

Definition 3.4 A *Metric Abstract Elementary Class* (MAEC), on a fixed similarity type $\mathcal{L}(\mathcal{K})$, is a class $\mathcal{K}$ of $\mathcal{L}(\mathcal{K})$-metric structures provided with a partial order $\prec_{\mathcal{K}}$ such that the following hold:

1. The class $\mathcal{L}$ is closed under isomorphism:
   (a) For every $\mathcal{M} \in \mathcal{K}$ and every $\mathcal{L}(\mathcal{K})$-structure $\mathcal{N}$, if $\mathcal{M} \simeq \mathcal{N}$ then $\mathcal{N} \in \mathcal{K}$.
   (b) Let $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{K}$ and $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{K}$ be such that there exists $f_\ell : \mathcal{N}_\ell \simeq \mathcal{M}_\ell$ (for $\ell = 1, 2$) satisfying $f_1 \subseteq f_2$. Then $\mathcal{N}_1 \prec_{\mathcal{K}} \mathcal{N}_2$ implies that $\mathcal{M}_1 \prec_{\mathcal{K}} \mathcal{M}_2$.
2. For all $\mathcal{M}, \mathcal{N} \in \mathcal{K}$ if $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}$ then $\mathcal{M} \subseteq \mathcal{N}$.
3. Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{M}^*$ be $\mathcal{L}(\mathcal{K})$-structures. If $\mathcal{M} \subseteq \mathcal{N}$, $\mathcal{M} \prec_{\mathcal{K}} \mathcal{M}^*$ and $\mathcal{N} \prec_{\mathcal{K}} \mathcal{M}^*$, then $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}$.
4. There exists a cardinal $\mathrm{LS}(\mathcal{K}) \geq \aleph_0 + |\mathcal{L}(\mathcal{K})|$ such that for every $\mathcal{M} \in \mathcal{K}$ and for every $A \subseteq M$ there exists $\mathcal{N} \in \mathcal{K}$ such that $\mathcal{N} \prec_{\mathcal{K}} \mathcal{M}$, $N \supseteq A$ and $\mathrm{dens}(N) \leq |A| + \mathrm{LS}(\mathcal{K})$ (downward Löwenheim-Skolem).
5. (a) For every cardinal μ and every $\mathcal{N} \in \mathcal{K}$, if $\{\mathcal{M}_i \prec_{\mathcal{K}} \mathcal{N} \mid i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing and continuous (i.e., $i < j \Rightarrow \mathcal{M}_i \prec_{\mathcal{K}} \mathcal{M}_j$) then $\overline{\bigcup_{i<\mu} \mathcal{M}_i} \in \mathcal{K}$ and $\overline{\bigcup_{i<\mu} \mathcal{M}_i} \prec_{\mathcal{K}} \mathcal{N}$.
   (b) For every μ, if $\{\mathcal{M}_i \mid i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing (i.e., $i < j \Rightarrow \mathcal{M}_i \prec_{\mathcal{K}} \mathcal{M}_j$) and continuous then $\overline{\bigcup_{i<\mu} \mathcal{M}_i} \in \mathcal{K}$ and for every $j < \mu$, $\mathcal{M}_j \prec_{\mathcal{K}} \overline{\bigcup_{i<\mu} \mathcal{M}_i}$.

   Here, $\overline{\bigcup_{i<\mu} \mathcal{M}_i}$ denotes the completion of $\bigcup_{i<\mu} \mathcal{M}_i$ (Tarski-Vaught chain).

Definition 3.5 Let (H, Γ_Q) be a structure as described in Definition 3.2. Let $\mathcal{L}$ the similarity type of (H, Γ_Q). We define $\mathcal{K}_{(H,\Gamma_Q)}$ to be the following class:

$$\mathcal{K}_{(H,\Gamma_Q)} := \{(H', \Gamma_{Q'}) \mid (H', \Gamma_{Q'}) \text{ is an } \mathcal{L}\text{-Hilbert space operator structure and } Q' \sim_\sigma Q\}$$

We define the relation $\prec_{\mathcal{K}}$ in $\mathcal{K}_{(H,\Gamma_Q)}$ by:

$$(H_1, \Gamma_{Q_1}) \prec_{\mathcal{K}} (H_2, \Gamma_{Q_2}) \text{ if and only if } H_1 \subseteq H_2 \text{ and } Q_1 \subseteq Q_2$$

Theorem 3.6 *The class $\mathcal{K}_{(H,\Gamma_Q)}$ is a MAEC.*

Proof. Condition 1(a) is clear by Lemma 3.3; conditions 1(b), 2, and 3 are clear. We consider condition 4. and claim that $\mathrm{LS}(\mathcal{K}) \leq 2^{2^{\aleph_0}}$. We first prove the following claim:

Claim 3.7 *If $(H', \Gamma_{Q'}) \in \mathcal{K}_{(H,\Gamma_Q)}$, there is a $(H'', \Gamma_{Q''}) \prec (H', \Gamma_{Q'})$ such that $(H'', \Gamma_{Q''}) \in \mathcal{K}$ and $|H''| \leq 2^{2^{\aleph_0}}$.*

Proof. By Corollary 2.40, there is a set $G' \subseteq H'$ such that $H' = H_{\mathrm{d}} \oplus \bigoplus_{v \in G'} H'_v$. Since there are at most $2^{2^{\aleph_0}}$ many Borel measures, there is a $G'' \subseteq G'$ such that $|G''| \leq 2^{2^{\aleph_0}}$ and for every $v \in G'$ there is a $w \in G''$ such that $\mu_v = \mu_w$. Take $H'' = H_{\mathrm{d}} \oplus \bigoplus_{v \in G''} H'_v$ and $Q'' := Q' \restriction H''$.

We have that Q'' is closed since H'' is a closed subset of H' and so is the graph of Q''. Then $(H'', \Gamma_{Q''}) \in \mathcal{K}_{(H,\Gamma_Q)}$, $(H'', \Gamma_{Q''}) \prec (H', \Gamma_{Q'})$ and $|H''| \leq 2^{2^{\aleph_0}}$. □

Now, let $(H', \Gamma_{Q'}) \in \mathcal{K}$ and $A \subseteq H'$. Let G' be as in Corollary 2.40 and let $(H'', \Gamma_{Q''})$ be as in Claim 3.7. Since $A \subseteq H_{\mathrm{d}} \oplus \bigoplus_{v \in G''} H'_v$, there is a $G_A \subseteq G''$, with $|G_A| \leq |A|\aleph_0$, such that $A \subseteq H_{\mathrm{d}} \oplus \bigoplus_{v \in G_A} H'_v$.

Let $\hat{H} := H_{\mathrm{d}} \oplus \bigoplus_{v \in G_A \cup G''} H'_v$ and $Q'' := Q' \restriction \hat{H}$. We have that Q'' is closed since $\hat{H}$ is a closed subset of H' and so is the graph of Q''. Then $(\hat{H}, \Gamma_{\hat{Q}}) \in \mathcal{K}_{(H,\Gamma_Q)}$, $(\hat{H}, \Gamma_{\hat{Q}}) \prec (H', \Gamma_{Q'})$, $A \subseteq \hat{H}$ and $|\hat{H}| \leq |A| + 2^{2^{\aleph_0}}$. This finishes the proof of condition 4.

Finally, we consider show the Tarski-Vaught chain property. To see condition 5(a), suppose κ is a regular cardinal and $(\hat{H}, \Gamma_{\hat{Q}}) \in \mathcal{K}_{(H,\Gamma_Q)}$. Let $(H_i, \Gamma_{Q_i})_{i<\kappa}$ a $\prec_{\mathcal{K}}$ increasing sequence such that $(H_i, \Gamma_{Q_i}) \prec_{\mathcal{K}} (\hat{H}, \Gamma_{\hat{Q}})$ for all $i < \kappa$. Then, for all $i < \kappa (H_{i+1}, \Gamma_{Q_{i+1}}) = (H_i, \Gamma_{Q_i}) \oplus (H'_i, \Gamma_{Q'_i})$, where H'_i is a Hilbert space and Q'_i is a (possibly unbounded) closed selfadjoint operator such that $\sigma_{\mathrm{d}}(Q'_i) = \varnothing$ and $\sigma_{\mathrm{e}}(Q'_i) \subseteq \sigma_{\mathrm{e}}(\hat{Q})$. Then $\overline{\bigcup_{i<\kappa}}(H_i, \Gamma_{Q_i}) = (H_0, \Gamma_{Q_0}) \bigoplus_{i<\kappa}(H'_i, \Gamma_{Q'_i})$. Since $(H_i, \Gamma_{Q_i}) \prec_{\mathcal{K}} (\hat{H}, \Gamma_{\hat{Q}})$, $\overline{\bigcup_{i<\kappa}}(H_i, \Gamma_{Q_i}) \prec_{\mathcal{K}} (\hat{H}, \Gamma_{\hat{Q}})$.

Condition 5(b) is clear from the argument for condition 5(a). □

From now on, the relation $\prec_{\mathcal{K}}$ in $\mathcal{K}_{(H,\Gamma_Q)}$ will be denoted as $\prec$.

Definition 3.8 Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be a MAEC and let $\mathcal{M}, \mathcal{N} \in \mathcal{K}$ be two structures. An emdedding $f : \mathcal{M} \to \mathcal{N}$ such that $f(\mathcal{M}) \prec_{\mathcal{K}} \mathcal{N}$ is called a $\mathcal{K}$-*embedding.* A MAEC $\mathcal{K}$ has the *Joint Embedding Property* (JEP) if for any $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{K}$ there are $\mathcal{N} \in \mathcal{K}$ and a $\mathcal{K}$-embeddings $f : \mathcal{M}_1 \to \mathcal{N}$ and $g : \mathcal{M}_2 \to \mathcal{N}$.

Theorem 3.9 *The MAEC $\mathcal{K}_{(H,\Gamma_Q)}$ has the JEP.*

Proof. Let $(H_1, \Gamma_{Q_1}), (H_1, \Gamma_{Q_2}) \in \mathcal{K}_{(H,\Gamma_Q)}$. Without loss of generality, we can assume that $H_1 \cap H_2 = \varnothing$. By Corollary 2.40, there are sets $G_1 \subseteq H_1$ and $G_2 \subseteq H_2$ such that $H_1 = H_{\mathrm{d}} \oplus \bigoplus_{v \in G_1}(H_1)_v$ and $H_2 = H_{\mathrm{d}} \oplus \bigoplus_{v \in G_2}(H_2)_v$.

Let

$$\hat{H} = H_{\mathrm{d}} \oplus \bigoplus_{v \in G_1}(H_1)_v \oplus \bigoplus_{v \in G_2}(H_2)_v$$

and

$$\hat{Q} := (Q_1 \restriction H_{\mathrm{d}}) \oplus \left(\bigoplus_{v \in G_1}(Q_1 \restriction (H_1)_v)\right) \oplus \left(\bigoplus_{v \in G_2}(Q_2 \restriction (H_2)_v)\right)$$

then, $\mathrm{Id}_{H_{\mathrm{d}}} \oplus \bigoplus_{v \in G_1} \mathrm{Id}_{(H_1)_v}$ and $\mathrm{Id}_{H_{\mathrm{d}}} \oplus \bigoplus_{v \in G_2} \mathrm{Id}_{(H_2)_v}$ are respective $\mathcal{K}_{(H,\Gamma_Q)}$-embeddings from (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}) to $(\hat{H}, \Gamma_{\hat{Q}})$. □

Definition 3.10 A MAEC $\mathcal{K}$ has the *Amalgamation Property* (AP) if for any $\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2 \in \mathcal{K}$ such that $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}_1$ and $\mathcal{M} \prec_{\mathcal{K}} \mathcal{N}_2$, there are $\mathcal{M}' \in \mathcal{K}$ and a $\mathcal{K}$-embeddings $f : \mathcal{N}_1 \to \mathcal{M}'$ and $g : \mathcal{N}_2 \to \mathcal{M}'$ such that $f(\mathcal{N}_1), g(\mathcal{N}_2) \prec_{\mathcal{K}} \mathcal{M}'$. and $f \restriction \mathcal{M} = g \restriction \mathcal{M}$.

Theorem 3.11 *The MAEC $\mathcal{K}_{(H,\Gamma_Q)}$ has the AP.*

Proof. Let (H_1, Γ_{Q_1}), (H_2, Γ_{Q_2}) and $(H_3, \Gamma_{Q_3}) \in \mathcal{K}_{(H,\Gamma_Q)}$ be such that $(H_1, \Gamma_{Q_1}) \prec (H_2, \Gamma_{Q_2})$ and $(H_1, \Gamma_{Q_1}) \prec (H_3, \Gamma_{Q_3})$. By Corollary 2.40, there are sets $G_1 \subseteq H_1$, $G_2 \subseteq H_2$ and $G_3 \subseteq H_3$ such that:

$$H_1 = H_{\mathrm{d}} \oplus \bigoplus_{v \in G_1}(H_1)_v,$$

$$H_2 = H_{\mathrm{d}} \oplus \bigoplus_{v \in G_1} (H_1)_v \oplus \bigoplus_{v \in G_2} (H_2)_v,$$

$$H_3 = H_{\mathrm{d}} \oplus \bigoplus_{v \in G_1} (H_1)_v \oplus \bigoplus_{v \in G_3} (H_3)_v.$$

Let

$$H_4 := H_{\mathrm{d}} \oplus \bigoplus_{v \in G_1} (H_1)_v \oplus \bigoplus_{v \in G_2} (H_2)_v \oplus \bigoplus_{v \in G_3} (H_3)_v$$

and

$$Q_4 := (Q_1 \restriction H_{\mathrm{d}}) \oplus \left(\bigoplus_{v \in G_1} (Q_1 \restriction (H_1)_v) \right) \oplus \left(\bigoplus_{v \in G_2} (Q_2 \restriction (H_2)_v) \right) \oplus \left(\bigoplus_{v \in G_3} (Q_3 \restriction (H_3)_v) \right).$$

Then $(H_4, \Gamma_{Q_4}) \in \mathcal{K}_{(H,\Gamma_Q)}$ and $\mathrm{Id}_{H_{\mathrm{d}}} \oplus \bigoplus_{v \in G_1} \mathrm{Id}_{(H_1)_v} \oplus \bigoplus_{v \in G_2} \mathrm{Id}_{(H_2)_v}$, $\mathrm{Id}_{H_{\mathrm{d}}} \oplus \bigoplus_{v \in G_1} \mathrm{Id}_{(H_1)_v} \oplus \bigoplus_{v \in G_3} \mathrm{Id}_{(H_3)_v}$ are respective $\mathcal{K}_{(H,\Gamma_Q)}$-embeddings from (H_2, Γ_{Q_2}) and (H_3, Γ_{Q_3}) to (H_4, Γ_{Q_4}). □

For (H_1, Γ_{Q_1}), (H_2, Γ_{Q_2}) and (H_3, Γ_{Q_3}) as in Theorem 3.11, we denote by

$$(H_2, \Gamma_{Q_2}) \bigvee_{(H_1, \Gamma_{Q_1})} (H_3, \Gamma_{Q_3}) := (H_2 \vee_{H_2} H_3, \Gamma_{Q_2 \vee_{Q_1} Q_3})$$

the *amalgamation* of (H_2, Γ_{Q_2}) and (H_3, Γ_{Q_3}) over (H_1, Γ_{Q_1}) as described in Theorem 3.11.

Definition 3.12 For $\mathcal{M}_1$, $\mathcal{M}_2 \in \mathcal{K}$, $A \subseteq \mathcal{M}_1 \cap \mathcal{M}_2$ and $(a_i)_{i<\alpha} \subseteq \mathcal{M}_1$, $(b_i)_{i<\alpha} \subseteq \mathcal{M}_2$, we say that $(a_i)_{i<\alpha}$ and $(b_i)_{i<\alpha}$ have the same *Galois type over A* in $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively, $(\mathrm{gatp}_{\mathcal{M}_1}((a_i)_{i<\alpha}/A) = \mathrm{gatp}_{\mathcal{M}_2}((b_i)_{i<\alpha}/A))$, if there are $\mathcal{N} \in \mathcal{K}$ and $\mathcal{K}$-embeddings $f : \mathcal{M}_1 \to \mathcal{N}$ and $g : \mathcal{M}_2 \to \mathcal{N}$ such that $f(a_i) = g(b_i)$ for every $i < \alpha$ and $f \restriction A \equiv g \restriction A \equiv \mathrm{Id}_A$, where Id_A is the identity on A.

Theorem 3.13 *Let* $v \in (H_1, \Gamma_{Q_1})$*,* $w \in (H_2, \Gamma_{Q_2})$ *and* $G \subseteq H_1 \cap H_2$ *such that* $(H_G, \Gamma_{Q_G}) \in \mathcal{K}_{(H,\Gamma_Q)}$*,* $(H_G, \Gamma_{Q_G}) \prec (H_1, \Gamma_{Q_1})$*,* $(H_G, \Gamma_{Q_G}) \prec (H_2, \Gamma_{Q_2})$*. Then* $\mathrm{gatp}_{(H_1,\Gamma_{Q_1})}(v/G) = \mathrm{gatp}_{(H_2,\Gamma_{Q_2})}(w/G)$ *if and only if* $P_G v = P_G w$ *and* $\mu_{P_{G^\perp} v} = \mu_{P_{G^\perp} w}$*.*

Proof. "⇒": Suppose $\mathrm{gatp}_{(H_1,\Gamma_{Q_1})}(v/G) = \mathrm{gatp}_{(H_2,\Gamma_{Q_2})}(w/G)$ and let $v' := P_{G^\perp} v$ and $w' := P_{G^\perp} w$. Then, by Definition 3.12, there exists $(H_3, \Gamma_{Q_3}) \in \mathcal{K}_{(H,\Gamma_Q)}$ and $\mathcal{K}_{(H,\Gamma_Q)}$-embeddings $U_1 : (H_1, \Gamma_{Q_1}) \to (H_3, \Gamma_{Q_3})$ and $U_2 : (H_2, \Gamma_{Q_2}) \to (H_3, \Gamma_{Q_3})$ such that $U_1 v = U_2 w$ and $U_1 \restriction G \equiv U_2 \restriction G \equiv \mathrm{Id}_G$, where Id_G is the identity on G. Since $v = P_G v + P_{G^\perp} v$, $w = P_G w + P_{G^\perp} w$ and $U_1 \restriction G \equiv U_2 \restriction G \equiv \mathrm{Id}_G$, we have that $U_1 P_G v = P_G v$ and $U_2 P_G w = P_G w$. Since U_1 and U_2 are embeddings, $\mu_{v'} = \mu_{U_1 v'} = \mu_{U_2 w'} = \mu_{w'}$.

"⇐": Let $v' := P_{G^\perp} v$ and $w' := P_{G^\perp} w$. Suppose $\mu_{v'} = \mu_{w'}$, then $\mu_{v'_{\mathrm{e}}} = \mu_{w'_{\mathrm{e}}} L^2(\mathbb{R}, \mu_{v'_{\mathrm{e}}}) = L^2(\mathbb{R}, \mu_{w'_{\mathrm{e}}})$. Let $\mu := \mu_{v'_{\mathrm{e}}} = \mu_{w'_{\mathrm{e}}}$. Also, let $\hat{H} := (H_1 \vee_{H_G} H_2) \oplus L^2(\mathbb{R}, \mu)$ and let $\hat{Q} := (Q_1 \vee_{Q_G} Q_2) \oplus M_{f_\mu}$ be as in the multiplication form of the Spectral Theorem. Let $U_1 : (H_1, \Gamma_{Q_1}) \to (\hat{H}, \hat{Q})$ be the $\mathcal{K}_{(H,\Gamma_Q)}$-embedding acting on $H_{v'}^\perp$ into $H_{v'}^\perp \vee H_{w'}^\perp$ as in the AP, and acting on $H_{v'}$ as in Fact 2.38. Define $U_2 : (H_2, \Gamma_{Q_2}) \to (\hat{H}, \hat{Q})$ in the same way. Then, we have completed the conditions to show that

$$\mathrm{gatp}_{(H_1,\Gamma_{Q_1})}(v/G) = \mathrm{gatp}_{(H_2,\Gamma_{Q_2})}(w/G).$$

□

Definition 3.14 A MAEC $\mathcal{K}$ is said to be *homogeneous* if whenever $\mathcal{M}, \mathcal{N} \in \mathcal{K}$ and $(a_i)_{i<\alpha} \subseteq \mathcal{M}$, $(b_i)_{i<\alpha} \subseteq \mathcal{N}$ are such that for all $n < \omega$ and $i_0, \dots, i_{n-1} < \alpha$

$$\mathrm{gatp}_{\mathcal{M}}(a_{i_0}, \dots, a_{i_{n-1}}/\varnothing) = \mathrm{gatp}_{\mathcal{N}}(b_{i_0}, \dots, b_{i_{n-1}}/\varnothing),$$

then we have that

$$\mathrm{gatp}_{\mathcal{M}}((a_i)_{i<\alpha}/\varnothing) = \mathrm{gatp}_{\mathcal{N}}((b_i)_{i<\alpha}/\varnothing).$$

Theorem 3.15 *The MAEC* $\mathcal{K}_{(H,\Gamma_Q)}$ *is homogeneous.*

Proof. Let (H_1, Γ_{Q_1}), $(H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H,\Gamma_Q)}$ and $(v_i)_{i<\alpha} \subseteq H_1$, $(w_i)_{i<\alpha} \subseteq H_2$ be such that for all $n < \omega$ and $i_0, \ldots, i_{n-1} < \alpha$

$$\mathrm{gatp}_{(H_1,\Gamma_{Q_1})}(v_{i_0}, \ldots, v_{i_{n-1}}/\varnothing) = \mathrm{gatp}_{(H_2,\Gamma_{Q_2})}(w_{i_0}, \ldots, w_{i_{n-1}}/\varnothing)$$

We can use Gram-Schmidt-like process to get orthonormal sequences. So, without loss of generality, we can assume that for all $i < \alpha v_i \in (H_1)_{\mathrm{e}}$, $w_i \in (H_2)_{\mathrm{e}}$ and for every $i \neq j < \alpha$, $v_i \perp v_j$ and $w_i \perp w_j$. For $i < \alpha$, let $\mu_i := \mu_{v_i} = \mu_{w_i}$, which agree by Theorem 3.13, since for all $i < \alpha \mathrm{gatp}_{(H_1,\Gamma_{Q_1})}(v_i/\varnothing) = \mathrm{gatp}_{(H_2,\Gamma_{Q_2})}(w_i/\varnothing)$. Also, let

$$\hat{H} := (H_1 \vee_{\varnothing} H_2) \oplus \bigoplus_{i<\alpha} L^2(\mathbb{R}, \mu_i)$$

and

$$\hat{Q} := (Q_1 \vee_{\varnothing} Q_2) \oplus \bigoplus_{i<\alpha} M_{f_{\mu_i}}$$

be as in the multiplication form of the Spectral Theorem. Let $U_1 : (H_1, \Gamma_{Q_1}) \to (\hat{H}, \Gamma_{\hat{Q}})$ be the $\mathcal{K}_{(H,\Gamma_Q)}$-embedding acting on $H^{\perp}_{(v_i)_{i<\alpha}}$ into $H^{\perp}_{(v_i)_{i<\alpha}} \vee H^{\perp}_{(w_i)_{i<\alpha}}$ as in the AP, and acting on $H_{(v_i)_{i<\alpha}}$ as in Fact 2.38. Define $U_2 : (H_2, \Gamma_{Q_2}) \to (\hat{H}, \hat{Q})$ in the same way. Then we have completed the conditions to show that

$$\mathrm{gatp}_{(H_1,\Gamma_{Q_1})}((v_i)_{i<\alpha}/\varnothing) = \mathrm{gatp}_{(H_2,\Gamma_{Q_2})}((w_i)_{i<\alpha}/\varnothing).$$

□

Theorem 3.16 ([17, Theorem 1.13]) *Let $(\mathcal{K}, \prec_{\mathcal{K}})$ a MAEC on a similarity type $\mathcal{L}$ satisfying JEP, AP and homogeneity. Let $\kappa > |\mathcal{L}| + \mathrm{LS}(\mathcal{K})$, then there is $\mathfrak{M} \in \mathcal{K}$ such that*

1. *$\mathfrak{M}$ is κ-universal, i.e., for all $\mathcal{M} \in \mathcal{K}$ such that $|\mathcal{M}| < \kappa$, there is a $\mathcal{K}$ embedding $f : \mathcal{M} \to \mathfrak{M}$; and*
2. *$\mathfrak{M}$ is κ-homogeneous, i.e., if $(a_i)_{i<\alpha}$, $(b_i)_{i<\alpha} \subseteq \mathfrak{M}$ are such that for all $n < \omega$ and $i_0, \ldots, i_{n-1} < \alpha$*

$$\mathrm{gatp}_{\mathfrak{M}}(a_{i_0}, \ldots, a_{i_{n-1}}/\varnothing) = \mathrm{gatp}_{\mathfrak{M}}(b_{i_0}, \ldots, b_{i_{n-1}}/\varnothing)$$

then there is an automorphism f of $\mathfrak{M}$ such that $f(a_i) = b_i$ for all $i < \alpha$.

If in the previous theorem, κ is a cardinal greater than the density of any structure in $\mathcal{K}$ that we want to study, the structure $\mathfrak{M}$ is called a *monster model*.

Let κ be as above, and let $\mathbb{M}(\mathbb{R})$ the set of all regular Borel meaures on $\mathbb{R}$ whoose support is disjoint from $\sigma_{\mathrm{p}}(Q)$. Then the structure $(\tilde{H}_\kappa, \Gamma_{\tilde{Q}_\kappa})$ where

$$\tilde{H} = H_{\mathrm{d}} \oplus \bigoplus_{\kappa} \left(\bigoplus_{\mu\in\mathbb{M}} L^2(\mathbb{R}, \mu) \right)$$

and

$$\tilde{Q} = (Q \restriction H_{\mathrm{d}}) \oplus \bigoplus_{\kappa} \left(\bigoplus_{\mu\in\mathbb{M}} M_{f_\mu} \right)$$

works as a monster model for $\mathcal{K}_{(H,\Gamma_Q)}$. This can be easily proven from the proofs of JEP, AP and homogeneity of $\mathcal{K}_{(H,\Gamma_Q)}$.

Definition 3.17 Let $\mathcal{K}$ be a MAEC that satisfies the JEP, AP and homogeneity. Let $\mathfrak{M}$ be a monster model for $\mathcal{K}$. Then $\mathcal{K}$ is said to have the *continuity of types property* if whenever $A \subseteq \mathfrak{M}$ and $(b_i)_{i<\omega}$ is a convergent sequence with limit $b = \lim_{n\to\infty} b_i$ such that $\mathrm{gatp}(b_i/A) = \mathrm{gatp}(b_j/A)$ for all $i, j < \omega$, then $\mathrm{gatp}(b/A) = \mathrm{gatp}(b_i/A)$ for all $i < \omega$.

Theorem 3.18 *The MAEC $\mathcal{K}_{(H,\Gamma_Q)}$ has the continuity of types property.*

Proof. Let $G \subseteq \tilde{H}$ be small and $(v_i)_{i<\omega} \subseteq \tilde{H}$ a sequence such that $\lim_{i\to\infty} v_i = v$ and $\mathrm{gatp}(v_i/G) = \mathrm{gatp}(v_j/G)$ for all $i, j < \omega$. Then by Theorem 3.13, $P_G v_i = P_G v_j$ and $\mathrm{gatp}(P_{G^\perp} v_i/\varnothing) = \mathrm{gatp}(P_{G^\perp} v_j/\varnothing)$ for all $i, j < \omega$. If $\lim_{i\to\infty} v_i = v$, it is clear that $P_G v_i = P_G v$ for all $i < \omega$. So it is enough to prove the theorem for the case $G = \varnothing$.

Suppose $\lim_{i\to\infty} v_i = v$ and $\mathrm{gatp}(v_i/\varnothing) = \mathrm{gatp}(v_j/\varnothing)$ for all $i, j < \omega$. By Theorem 3.13, this means that $\mu_i = \mu_j$ for all $i, j < \omega$. Let $\mu := \mu_i$ and $E \subseteq \mathbb{R}$ be a Borel set. Then

$$\begin{aligned}\langle \chi_E(Q)v \mid v\rangle &= \langle \chi_E(Q) \lim_{i\to\infty} v_i \mid \lim_{i\to\infty} v_i\rangle = \lim_{i\to\infty} \langle \chi_E(Q)v_i \mid v_i\rangle \\ &= \lim_{i\to\infty} \mu_i(E) = \lim_{i\to\infty} \mu(E) = \mu(E).\end{aligned}$$

Again by Theorem 3.13, $\mathrm{gatp}(v_i/\varnothing) = \mathrm{gatp}(v/\varnothing)$ for all $i < \omega$. □

In the equalities used in the proof of Theorem 3.18, we can exchange the limit with $\chi(Q)$ because $\chi(Q)$ is a bounded (and therefore continuous) operator.

4 Definable and algebraic closures

In this section we give a characterization of definable and algebraic closures.

Definition 4.1 Let $\mathcal{K}$ be a MAEC with JEP and AP. Let $\mathfrak{M}$ be the monster model in $\mathcal{K}$ and let $A \subseteq \mathfrak{M}$ be a small subset. Then the definable closure and the algebraic closure of A are the sets

$$\mathrm{dcl}(A) := \{m \in \mathfrak{M} \mid \text{for all automorphisms } F \text{ of } \mathfrak{M} \text{ that fix } A \text{ pointwise, we have that } Fm = m\}$$

and

$$\mathrm{acl}(A) := \{m \in \mathfrak{M} \mid \text{the orbit under } \mathrm{Aut}(\mathfrak{M}/A) \text{ is compact}\},$$

respectively.

Recall that $\mathrm{Aut}(\mathfrak{M}/A)$ is the group of automorphisms of $\mathfrak{M}$ that fix A pointwise.

Theorem 4.2 *Let $G \subseteq \tilde{H}$. Then* $\mathrm{dcl}(G) = \tilde{H}_G$.

Proof. "$\mathrm{dcl}(G) \subseteq \tilde{H}_G$": Let $v \notin \tilde{H}_G$. Then $P_{G^\perp} v \neq 0$. Let $(H', \Gamma_{Q'}) \in \mathcal{K}_{(H,\Gamma_Q)}$ be a small structure containing v. Let $(H'', \Gamma_{Q''}) \in \mathcal{K}_{(H,\Gamma_Q)}$ be a structure containig $H' \oplus L^2(\mathbb{R}, \mu_{P_{G^\perp} v_e})$. Let $w := P_G v + (1)_{\mu_{P_{G^\perp} v_e}} \in H''$. Then $\mathrm{gatp}(v/G) = \mathrm{gatp}(w/G)$, but $v \neq w$. Therefore $v \notin \mathrm{dcl}(G)$.

"$\tilde{H}_G \subseteq \mathrm{dcl}(G)$": Let $v \in G$, let h be a bounded Borel function on $\mathbb{R}$, let $U \in \mathrm{Aut}(\tilde{H}, \tilde{Q}/G)$ and let $(H', \Gamma_{Q'})$ a small structure containg G. Then, by Lemma 3.3, $Uh(Q')v = h(Q')Uv = h(Q')v$, and $v \in \mathrm{dcl}(G)$. □

Lemma 4.3 *Let $v \in \tilde{H}$. If v is an eigenvector corresponding to some $\lambda \in \sigma_{\mathrm{d}}(Q)$ then v is algebraic over $\varnothing$.*

Proof. We have that $\lambda \in \sigma_{\mathrm{d}}(Q)$ if and only if λ is isolated in $\sigma(Q)$ with finite dimensional eigenspace $\tilde{H}_\lambda$. So any automorphism can only send $\tilde{H}_\lambda$ onto $\tilde{H}_\lambda$ and the orbit of v under such automorphism can only be compact. □

Lemma 4.4 *Let $v \in \tilde{H}$ be such that $v = \sum v_k$ where each v_k is an eigenvector for some $\lambda_k \in \sigma_{\mathrm{d}}(Q)$. Then v is algebraic over $\varnothing$.*

Proof. Given that $\|v_k\| \to 0$ when $k \to \infty$, the orbit of v under all the automorphisms is a Hilbert cube which is compact. □

Theorem 4.5 *We have that* $\mathrm{acl}(\varnothing) = H_{\mathrm{d}}$.

Proof. That $\mathrm{acl}(\varnothing) \subseteq H_{\mathrm{d}}$ is a consequence of Lemma 4.4. For the converse, suppose $v \in \tilde{H}$ such that $v_e \neq 0$. Let η be an uncountable small cardinal and let $F := \bigoplus_\eta L^2(\mathbb{R}, \mu_{v_e})$. Any structure in $\mathcal{K}_{(H,\Gamma_Q)}$ containing G will have η different realizations of $\mathrm{gatp}(v/\varnothing)$. Therefore $v \notin \mathrm{acl}(\varnothing)$. □

Theorem 4.6 *Let $G \subseteq \tilde{H}$. Then* $\mathrm{acl}(G)$ *is closed Hilbert subspace generated by the union of* $\mathrm{dcl}(G)$ *with* $\mathrm{acl}(\varnothing)$.

P r o o f . Let E be the space $\mathrm{acl}(\emptyset) + \mathrm{dcl}(G)$. We have that $\mathrm{acl}(\emptyset) \subseteq \mathrm{acl}(G)$ and $\mathrm{dcl}(G) \subseteq \mathrm{acl}(G)$ so $E \subseteq \mathrm{acl}(G)$. If $v \notin E$, then $P_E^{\perp} v \neq 0$. Let η be an uncountable small cardinal and let $F := \bigoplus_\eta L^2(\mathbb{R}, \mu_{(P_E^{\perp} v)_e})$. Any structure in $\mathcal{K}_{(H,\Gamma_Q)}$ containing G will have η different realizations of $\mathrm{gatp}(v/G)$. Therefore, $v \notin \mathrm{acl}(A)$. □

5 Perturbations

In this section, we define a system of perturbations for $\mathcal{K}_{(H,\Gamma_Q)}$ and show that $\mathcal{K}_{(H,\Gamma_Q)}$ is separably categorical up to this system of perturbations.

Definition 5.1 Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be a MAEC. A class $(\mathbb{F}_\varepsilon)_{\varepsilon \geq 0}$ collections of bijective mappings between members of $\mathcal{K}$ is said to be a *system of perturbations* for $(\mathcal{K}, \prec_{\mathcal{K}})$ if it satisfies the following conditions:

1. If $\delta < \varepsilon$, then $\mathbb{F}_\delta \subseteq \mathbb{F}_\varepsilon$; furthermore, $\mathbb{F}_0 = \bigcap_{\varepsilon > 0} \mathbb{F}_\varepsilon$ and $\mathbb{F}_0$ is exactly the collection of real isomorphisms of structures in $\mathcal{K}$.
2. If $f : \mathcal{M} \to \mathcal{N}$ is in $\mathbb{F}_\varepsilon$, then f is a e^ε-bi lipschitz mapping with respect to the metric, i.e., $e^{-\varepsilon} d(x, y) \leq d(f(x), f(y)) \leq e^\varepsilon d(x, y)$ for all $x, y \in M$.
3. If $f \in \mathbb{F}_\varepsilon$ then $f^{-1} \in \mathbb{F}_\varepsilon$.
4. If $f \in \mathbb{F}_\varepsilon$, $g \in \mathbb{F}_\delta$, and $\mathrm{dom}(g) = \mathrm{rng}(f)$ then $g \circ f \in \mathbb{F}_{\varepsilon+\delta}$.
5. If $(f_i)_{i<\alpha}$ is an increasing chain of ε-isomorphisms, i.e., $f_i \in \mathbb{F}_\varepsilon$, $f_i : \mathcal{M}_i \to \mathcal{N}_i$, $\mathcal{M}_i \prec_{\mathcal{K}} \mathcal{M}_{i+1}$, $\mathcal{N}_i \prec_{\mathcal{K}} \mathcal{N}_{i+1}$ and $f_i \subseteq f_{i+1}$ for every $i < \alpha$, then there is an ε-isomorphism $f : \overline{\bigcup_{i<\alpha} \mathcal{M}_i} \to \overline{\bigcup_{i<\alpha} \mathcal{N}_i}$ such that $f \restriction \mathcal{M}_i = f_i$ for all $i < \omega$.

If $(\mathbb{F}_\varepsilon)_{\varepsilon \geq 0}$ is a system of perturbations for $(\mathcal{K}, \prec_{\mathcal{K}})$, then $(\mathcal{K}, \prec_{\mathcal{K}}, (\mathbb{F}_\varepsilon)_{\varepsilon \geq 0})$ is called a *MAEC with perturbations.*

Definition 5.2 Let $\varepsilon > 0$. An *ε-perturbation* in $\mathcal{K}_{(H,\Gamma_Q)}$ is an unitary operator $U : H_1 \to H_2$ such that there are closed unbounded selfadjoint operators Q_1 and Q_2 defined on H_1 and H_2 respectively, such that

1. $(H_1, \Gamma_{Q_1}), (H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H,\Gamma_Q)}$,
2. $UD(Q_1) = D(Q_1)$,
3. the operator $Q_1 - U^{-1} Q_2 U$ can be extended to a bounded operator on H_1 with norm less or equal to ε, and
4. the operator $Q_2 - U Q_1 U^{-1}$ can be extended to a bounded operator on H_2 with norm less or equal to ε.

The class of all ε-perturbations in $\mathcal{K}_{(H,\Gamma_Q)}$ is denoted by $(\mathbb{F}_\varepsilon^{(H,\Gamma_Q)})_{\varepsilon \geq 0}$

Theorem 5.3 *The tuple $(\mathcal{K}_{(H,\Gamma_Q)}, \prec_{\mathcal{K}_{(H,\Gamma_Q)}}, (\mathbb{F}_\varepsilon^{(H,\Gamma_Q)})_{\varepsilon \geq 0})$ is a MAEC with perturbations.*

P r o o f . Items (1), (2) and (3) are clear. (4) follows from the triangle inequality. For (5), recall from the Tarski chain condition in Theorem 3.6 that $\overline{\bigcup_{i<\kappa} (H_i, \Gamma_{Q_i})} = H_0 \bigoplus_{i<\kappa} (H_i', \Gamma_{Q_i'})$. This with the fact that a direct sum of κ bounded operators with norm less than ε is still a bounded operator with norm less than ε. □

Definition 5.4 A MAEC with a system or perturbations $(\mathcal{K}, \prec_{\mathcal{K}}, (\mathbb{F}_\varepsilon)_{\varepsilon \geq 0})$ is said to be $\aleph_0$-*categorical up to the system of perturbations* $(\mathbb{F}_\varepsilon)_{\varepsilon \geq 0}$, if for all separable $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{K}$ and for all $\varepsilon > 0$, there is an $f_\varepsilon \in \mathbb{F}_\varepsilon$ such that $f_\varepsilon : \mathcal{M}_1 \to \mathcal{M}_2$.

Theorem 5.5 *The MAEC with a system of perturbations $(\mathcal{K}_{(H,\Gamma_Q)}, \prec_{\mathcal{K}_{(H,\Gamma_Q)}}, (\mathbb{F}_\varepsilon^{(H,\Gamma_Q)})_{\varepsilon \geq 0})$ is $\aleph_0$-categorical up to the system of perturbations.*

P r o o f . Let $(H_1, \Gamma_{Q_1}), (H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H,\Gamma_Q)}$ be separable. For each $\varepsilon > 0$, we build a structure $(H_\varepsilon, \Gamma_{Q_\varepsilon})$, an ε-isomorphism $V_\varepsilon : (H_1, \Gamma_{Q_1}) \to (H_\varepsilon, \Gamma_{Q_\varepsilon})$ and an ε-isomorphism $W_\varepsilon : (H_2, \Gamma_{Q_2}) \to (H_\varepsilon, \Gamma_{Q_\varepsilon})$. So, $V_\varepsilon W_\varepsilon^*$ is an 2ε-isomorphism between (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}). This shows that $\mathcal{K}_{(H,\Gamma_Q)}$ is $\aleph_0$-categorical up to the system of perturbations.

Now, let us go to the construction of the V_ε's: Let $\varepsilon > 0$ and let $(I_k)_{k \in \mathbb{Z}^+}$ be a family of disjoint connected subsets of $\mathbb{R}$ with diameter less than ε, which also cover $\sigma_e(Q)$. Let $(\lambda_k)_{k \in \mathbb{Z}^+} \subseteq \sigma(Q)$ be a set of inner points in each of the I_k's. Let $(H'_k, \Gamma_{Q'_k})$ be an $\aleph_0$-dimensional structure such that Q'_k acts on H'_k as λ_k times the identity. Given I_k, both $\chi_{I_k}(Q_1)H_1$ and H'_k are separable and infinite dimensional. Therefore, there is an isomorphism:

$$V_\varepsilon^k : (\chi_{I_k}(Q_1)H_1, \Gamma_{Q_1\restriction \chi_{I_k}(Q_1)H_1}) \to (H'_k, \Gamma_{Q'_k})$$

Now, let $\lambda_{d_1}, \dots, \lambda_{d_{n_\varepsilon}}$ be the (finite) set of discrete spectral values (isolated finite dimensional eigenvalues) not covered by $(I_k)_{k\in\mathbb{Z}^+}$. Let H_{d_i} be the eigenspace of λ_{d_i} and let n_{d_i} be the dimension of H_{d_i}. Let Q_i be the restriction of Q_1 to H_{d_i}.

Let

$$(H_\varepsilon, \Gamma_{Q_\varepsilon}) := \bigoplus_{i=1}^{n_\varepsilon}(H_{d_i}, \Gamma_{Q_{d_i}}) \oplus \bigoplus_{k\in\mathbb{Z}^+}(H'_k, \Gamma_{Q'_k})$$

and let

$$V_\varepsilon := \bigoplus_{i=1}^{n_\varepsilon} \mathrm{Id}_{H_{d_i}} \oplus \bigoplus_{k\in\mathbb{Z}^+} V_\varepsilon^k.$$

Given that $|x - \sum_{i=1}^{n_\varepsilon} \lambda_{d_i}\chi_{\{\lambda_{d_i}\}} - \sum_{k\in\mathbb{Z}^+}\lambda_k\chi_{I_k}| < \varepsilon$, we get that $\|Q_1 - V_\varepsilon^* Q_\varepsilon V_\varepsilon\| < \varepsilon$. So, we have completed the proof. □

Remark 5.6 Theorem 5.5 implies that any two separable structures (H_1, Γ_{Q_1}), $(H_2, \Gamma_{Q_2}) \in \mathcal{K}_{(H,\Gamma_Q)}$ are approximately unitarily equivalent.

6 CFO elementary equivalence and continuous $\mathcal{L}_{\omega_1,\omega}$ axiomatization

In this section we deal with continuous first order elementary equivalence for the structures of the type (H, Γ_Q).

Lemma 6.1 *For every bounded linear operator $S \in \mathrm{B}(H)$, definable in (H, Γ_Q), and for all v and $w \in H$, we have that $\|Sv - w\| \le (2 + \|S\|)\Gamma_S(v, w)$ where $\Gamma_S(v, w)$ denotes the distance to the graph of S.*

Proof. Let G_S be the Hilbert subspace of $H \times H$ given by $G_S := \{(v, Sv) \mid v \in H\}$ and let P_S be the projection $H \times H$ over G_S. If $(v', Sv') := P_S(v, w)$, then $\Gamma_S(v, w) = d[(v', Sv'), (v, w)]$. So, $\|Sv - w\| \le \Gamma_S(v, w) + d[(v', Sv'), (v, Sv)] \le \Gamma_S(v, w) + d(v', v) + d(Sv', Sv) \le \Gamma_S(v, w) + \Gamma_S(v, w) + \|S\|d(v', v) \le 2\Gamma_S(v, w) + \|S\|\Gamma_S(v, w) = (2 + \|S\|)\Gamma_S(v, w)$. □

Lemma 6.2 *For every bounded linear operator $S \in \mathrm{B}(H)$, definable in (H, Γ_Q), the following condition holds in (H, Γ_Q):*

$$\sup_v \sup_{w_1,w_2} \left(\left\| \frac{w_1 - w_2}{2} \right\| \dot{-} (2 + \|S\|) \frac{\Gamma_S(v, w_1) + \Gamma_S(v, w_2)}{2} \right) = 0.$$

Proof. Let $\bar{v}_1 := (v, w_1)$ and $\bar{v}_2 := (v, w_2)$, two pairs in $H \times H$. Then $\|w_1 - w_2\| \le \|Sv - w_1\| + \|Sv - w_2\| \le (2 + \|S\|)\Gamma_S(v, w_1) + (2 + \|S\|)\Gamma_S(v, w_2)$. □

Lemma 6.3 *For every closed linear operator S on H, definable in (H, Γ_Q), the following condition holds in (H, Γ_Q):*

$$\sup_{v_1,v_2,w_2,w_3} \left(\Gamma_S^2\left(\frac{v_1 + v_2}{2}, \frac{w_1 + w_2}{2}\right) \dot{-} \left(\frac{\Gamma_S(v_1, w_1) + \Gamma_S(v_2, w_2)}{2}\right)^2 \right) = 0.$$

Proof. Let $\bar{v}_1 := (v_1, w_1)$ and $\bar{v}_2 := (v_2, w_2)$, be two pairs in $H \times H$. Let $\bar{v}'_1 := (v'_1, w'_1)$ and let $\bar{v}'_2 := (v'_2, w'_2)$ be pairs in $H \times H$ such that $\Gamma_S(v'_1, w'_1) = \Gamma_S(v'_2, w'_2) = 0$. Then $d(\frac{\bar{v}_1+\bar{v}_2}{2}, \frac{\bar{v}'_1+\bar{v}'_2}{2}) \le \frac{d(\bar{v}_1,\bar{v}'_1)+d(\bar{v}_2,\bar{v}'_2)}{2}$. Notice that, since v'_1 and v'_2 belong to the domain of S, so does $\frac{v'_1+v'_2}{2}$. So,

$$\left(d\left(\frac{\bar{v}_1 + \bar{v}_2}{2}, \frac{\bar{v}'_1 + \bar{v}'_2}{2}\right)\right)^2 \le \left(\frac{d(\bar{v}_1, \bar{v}'_1) + d(\bar{v}_2, \bar{v}'_2)}{2}\right)^2.$$

Now, since S is closed, there exist $(\bar{v}_1^k := (v_1^k, w_1^k))_{k\in\mathbb{N}}$ and $(\bar{v}_2^k := (v_2^k, w_2^k))_{k\in\mathbb{N}}$, two sequences of pairs in $H \times H$ such that $\Gamma_S(v_1, w_1) = \lim_{k\to\infty} d[(v_1, w_1), (v_1^k, w_1^k)]$ and $\Gamma_S(v_2, w_2) = \lim_{k\to\infty} d[(v_2, w_2), (v_2^k, w_2^k)]$. Replacing in previous inequality and taking limits, we get the desired result. □

The next theorem is an adaptation of one developed by Argoty and Ben Yaacov in [4]:

Theorem 6.4 *Let h be a bounded (complex) Borel function on $\mathbb{R}$. Then $\Gamma_{h(Q)}$ is definable in (H, Γ_Q) if and only if $h \in \mathcal{C}(\sigma(Q), \mathcal{C})$.*

P r o o f . In this proof, $\prec$ will denote the usual notion of first order elementary substructure. Also, $(\hat{H}, \Gamma_{\hat{Q}})$ will denote a first order elementary extension of (H, Γ_Q) which is saturated and homogeneous.

"$\Rightarrow$": Suppose h is a bounded Borel function on $\mathbb{R}$ which is not continuous on $\sigma(Q)$ and such that $\Gamma_{h(Q)}$ is definable in in (H, Γ_Q). Let $\lambda_0 \in \sigma(Q)$ be a point of discontinuity of h. Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence in $\sigma(Q)$ and $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$ such that $\lim_{\mathcal{U}} \lambda_k = \lambda_0$ and such that $\lim_{\mathcal{U}} h(\lambda_k)$ exists but $\lim_U h(\lambda_k) \neq h(\lambda_0)$. There exist models $(\mathcal{H}_k, \Gamma_{Q_k}) \prec (\hat{H}, \Gamma_{\hat{Q}})$ and $v_k \in \mathcal{H}_k$ for $k \in \mathbb{N}$ such that $\mathcal{H}_k \models \Gamma_Q(v_k, \lambda_k v_k) = 0$. Let $\mathcal{H} = \Pi_{\mathcal{U}}\mathcal{H}_k$ and let $v = (v_k)_{\mathcal{U}} \in \mathcal{H}$. Then $(v_k)_{\mathcal{U}}$ is an eigenvector in $\mathcal{H}$ for the eigenvalue λ_0, and we have

$$h(\lambda_0)v = h(Q)(v) = h(Q)(v_k)_{\mathcal{U}} = (h(Q)v_k)_{\mathcal{U}} = (\lambda_k v_k)_{\mathcal{U}} = (\lim_U h(\lambda_k))(v_k)_{\mathcal{U}} = (\lim_U h(\lambda_k))v.$$

So $h(\lambda_0) = \lim_{\mathcal{U}} h(\lambda_k)$ which is a contradiction.

"$\Leftarrow$": Suppose $h \in \mathcal{C}(\sigma(Q), \mathcal{C})$. Then by the Stone-Weierstrass theorem h can be uniformly approximated by a sequence of polynomials over $\sigma(Q)$. These polynomials are translated into polynomials in Q. Such polynomials are definable, so $h(Q)$ is definable. □

Lemma 6.5 *If $\lambda \in \sigma(Q)$, λ is isolated if and only if $\Gamma_{\chi_{\{\lambda\}}(Q)}$ is definable in (H, Γ_Q).*

P r o o f . If $\lambda \in \sigma(Q)$, then $\chi_{\{\lambda\}}$ is continuous on $\sigma(Q)$ if and only if λ is isolated in $\sigma(Q)$. By Theorem 6.4, $\Gamma_{\chi_{\{\lambda\}}(Q)}$ is definable in (H, Γ_Q) if and only if $\chi_{\{\lambda\}}$ is continuous on $\sigma(Q)$. □

Lemma 6.6 *The following are equivalent:*

1. *$\lambda \in \sigma_{\mathrm{e}}(Q)$ and*
2. *for every $n \in \mathbb{N}$ and every bounded Borel function $h : \mathbb{R} \to \mathcal{C}$ such that h is continuous on $\sigma(Q)$ and $h(\lambda) \neq 0$, we have that*

$$\inf_{v_1 v_2 \cdots v_n} \inf_{w_1, w_2 \cdots w_n} \max_{i,j=1,\cdots,n} (|\langle w_i \mid w_j\rangle - \delta_{ij}|, \|h(Q)v_i - w_i\|) = 0 \tag{†}$$

holds in (H, Q).

P r o o f . "(1.)$\Rightarrow$(2.)": Suppose $\lambda \in \sigma_{\mathrm{e}}(Q)$ and let $h : \mathbb{R} \to \mathcal{C}$ be a bounded Borel function such that h is continuous on $\sigma(Q)$ and $h(\lambda) \neq 0$. Then there is an open set $V \subseteq \mathbb{R}$ such that $\lambda \in V$ and h does not have any zero in V. Even more, we can choose h such that there is an $M > 0$ such that $|h| > M$. Since $\lambda \in \sigma_{\mathrm{e}}(Q)$, the space $\chi_V(Q)H$ is infinite dimensional and since h does not have any zero in V, there is a function h^{-1} which is continuous on V, $h^{-1}h \equiv 1$ (in the multiplicative sense) on V and h^{-1} can be extended contiuously on $\mathbb{R}$. By the Functional Calculus Form of the Spectral Theorem, $h(Q)h^{-1}(Q) \equiv \mathrm{Id}_{\chi_V(Q)H}$ where $\mathrm{Id}_{\chi_V(Q)H}$ is the identity operator on $\chi_V(Q)H$. This implies that $h(Q)$ is invertible in $\chi_V(Q)H$ and therefore the dimension of $h(Q)\chi_V(Q)H = h(Q)H$ is infinite.

On the other hand, by Theorem 6.4, the condition in (†) can be expressed in continuous first order logic, and corresponds to the first order sentence:

$$\exists v_1 v_2 \cdots v_n \exists w_1 w_2 \cdots w_n \left(\langle w_i | w_j\rangle = \delta_{ij} \wedge h(Q)v_i = w_i\right),$$

where δ_{ij} is Kronecker's delta. This condition states that $h(Q)H$ has dimension greater than n.

"(2.)$\Rightarrow$(1.)": Suppose that for every $n \in \mathbb{N}$, and every bounded Borel function $h : \mathbb{R} \to \mathcal{C}$ such that f is continuous on $\sigma(Q)$ and $h(\lambda) \neq 0$, we have that (†) holds. Let $\varepsilon > 0$ and let h_n be a sequence of continuous functions on $\mathbb{R}$ that converge to $\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}$. By the Functional Calculus Form of the Spectral Theorem, $h_n(Q) \to \chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(Q)$ in the norm. Since $h_n(Q)H$ is infinite dimensional for all $n \in \mathbb{N}$, $\chi_{(\lambda-\varepsilon,\lambda+\varepsilon)}(Q)H$ is infinite dimensional. Since $\varepsilon > 0$ is arbitrary, by Theorem 2.26, $\lambda \in \sigma_{\mathrm{e}}(Q)$. □

Lemma 6.7 *If λ is a complex number. Then $\lambda \in \sigma_{\mathrm{d}}(Q)$ if and only if there exists $n \in \mathbb{N}$ such that the following condition exists in continuous first order logic and is true in (H, Γ_Q):*

$$\inf_{v_1 v_2 \cdots v_n} \sup_{w} \max \left(|\langle v_i | v_j \rangle - \delta_{ij}|, \Gamma_Q(v_i, \lambda v_i), \left\| \chi_{\{\lambda\}}(Q)w - \sum_{k=1}^{n} \langle w | v_i \rangle v_i \right\| \right) = 0 \qquad (\ddagger)$$

where δ_{ij} is Kronecker's delta.

P r o o f. By Lemma 6.5, λ is isolated in $\sigma(Q)$ if and only if $\Gamma_{\chi_{\{\lambda\}}(Q)}$ is definable in (H, Q). Then, (‡) can be expressed in continuous first order logic if and only if λ is isolated in $\sigma(Q)$. On the other hand, (‡) is a continuous first order condition corresponding to

$$\exists v_1 v_2 \cdots v_n \forall w \left(\left(\bigwedge_{i,j} \langle v_i | v_j \rangle = \delta_{ij} \right) \wedge \left(\bigwedge_{i=1}^{n} Q v_i = \lambda v_i \right) \wedge \chi_{\{\lambda\}}(Q)w = \sum_{k=1}^{m} \langle w | v_i \rangle v_i \right) = 0.$$

In particular, the statement

$$\chi_{\{\lambda\}}(Q)w = \sum_{k=1}^{m} \langle w | v_i \rangle v_i$$

means that the vectors $v_1, \cdots, v_n$ generate the eigenspace of λ. So the dimension of the eigenspace of λ is n. $\square$

Lemma 6.8 *If λ is a complex number, then $\lambda \notin \sigma(Q)$ if and only if for some $c > 0$ and for some continuous function $f : [0, 1] \to [0, 1]$ such that $f(0) = 0$, the following condition is true in (H, Γ_Q):*

$$\sup_{v} \sup_{w} ((c\|v\| \dot{-} \|w\|) \dot{-} f(\Gamma_Q(v, \lambda v + w))) = 0. \qquad (\#)$$

P r o o f. "$\Rightarrow$": Suppose $\lambda \notin \sigma(Q)$. By Fact 2.15 there exists $c > 0$ such that for every $v \in D(Q)$, $\|(Q - \lambda I)v\| \geq c\|v\|$. Given $r \in [0, 1]$, let $f(r) := \sup\{c\|v\| \dot{-} \|w\| \mid \Gamma_Q(v, \lambda v + w) = r\}$. The function f is well defined, since the set $\{c\|v\| \dot{-} \|w\| \mid \Gamma_Q(v, \lambda v + w) = r\}$ is bounded in $\mathbb{R}$ for all $r \in [0, 1]$, and is also continuous on $[0, 1]$. Now, $f(0) = sup\{c\|v\| \dot{-} \|w\| \mid \Gamma_Q(v, \lambda v + w) = 0\}$; the condition $\Gamma_Q(v, \lambda v + w) = 0$ means that $v \in D(Q)$ and $Qv = \lambda v + w$, which means that $w = Qv - \lambda v$. So, $w = R_\lambda v$ and by Theorem 2.15, $\|w\| \geq c\|w\|$ thus $c\|v\| \dot{-} \|w\| = 0$. Therefore $f(0) = 0$.

"$\Leftarrow$": Suppose now that (#) holds for some $c > 0$ and $f : [0, 1] \to [0, 1]$ continuous such that $f(0) = 0$. Then if $v \in D(Q)$ and $w := (Q - \lambda I)v$, $\Gamma_Q(v, \lambda v + w) = 0$ and since $f(0) = 0$, $f(\Gamma_Q(v, \lambda v + w)) = 0$. By (#), we have that $c\|v\| \dot{-} \|w\| = 0$ and therefore $c\|v\| \leq \|w\|$ what, by Theorem 2.15, means that $\lambda \in \varrho(Q)$. $\square$

The next theorem was remarked by Henson:

Theorem 6.9 *Let Q_1 and Q_2 be two closed (unbounded) self adjoint operators on the separable Hilbert space H. Then the following statements are equivalent:*

1. *The operators Q_1 and Q_2 are approximately unitarily equivalent.*
2. *The structures (H, Γ_{Q_1}) and (H, Γ_{Q_2}) are elementarily equivalent.*
3. *$Q_1 \sim_\sigma Q_1$.*

P r o o f. "(1)$\Rightarrow$(2)": Suppose that Q_1 and Q_2 are approximately unitarily equivalent. Then there exists a sequence of unitary operators U_n on H such that $\lim_{n\to\infty} U_n Q_1 U_n^* = Q_2$. Let $\mathbb{N}$ be an ultrafilter over $\mathbb{N}$ which contains the filter of cofinite subsets of $\mathbb{N}$. Let $(\hat{H}_1, \Gamma_{\hat{Q}_1}) = \Pi_{\mathbb{N}}(H, \Gamma_{U_n Q_1 U_n^*})$ and let $(\hat{H}_2, \Gamma_{\hat{Q}_2}) = \Pi_{\mathbb{N}}(H, \Gamma_{Q_2})$. It follows that $(\hat{H}_1, \Gamma_{\hat{Q}_1}) \simeq (\hat{H}_2, \Gamma_{\hat{Q}_2})$ and by the Keisler-Shelah Theorem, $(H, \Gamma_{Q_1}) \equiv (H, \Gamma_{Q_2})$.

"(2)$\Rightarrow$(3)" Suppose $(H, \Gamma_{Q_1}) \equiv (H, \Gamma_{Q_2})$. Since the relation $Q_1 \sim_\sigma Q_2$ can be written down as sets of conditions in continuous first order logic (cf. Lemmas 6.6, 6.7 and 6.8), we have that $Q_1 \sim_\sigma Q_2$.

"(3)$\Rightarrow$(1)": Suppose now that $Q_1 \sim_\sigma Q_2$. Then $(H, \Gamma_{Q_2}) \in \mathcal{K}_{(H_1, \Gamma_{Q_1})}$. By Theorem 5.5 and Remark 5.6, Q_1 and Q_2 are approximately unitarily equivalent. $\square$

Definition 6.10 Let $\mathsf{IHS}_{\sigma(Q)}$ the theory of Hilbert spaces together with the following conditions (1) to (9) in continuous $\mathcal{L}_{\omega_1\omega}$. Let $h : \sigma(Q) \to \mathcal{C}$ be a continuous bounded function and $\lambda \in \sigma_{\mathrm{d}}(Q)$, $\mu \in \sigma)\mathrm{e}(Q)$, and

$\nu \in \varrho(Q)$; let g be an arbitrary bounded Borel function $g : \mathbb{R} \to \mathcal{C}$ that is continuous on $\sigma(Q)$ and $g(\mu) \neq 0$ (cf. Lemma 6.6) and let c_ν and $f_\nu : [0, 1] \to [0, 1]$ be such that they satisfy the hypothesis of Lemma 6.8. We let $n_\lambda := \dim \chi_{\{\lambda\}}(Q)H$.

$$\sup_{v} \sup_{w_1, w_2} \left(\left\| \frac{w_1 - w_2}{2} \right\| \dot{-} (2 + \|h(Q)\|) \frac{\Gamma_{h(Q)}(v, w_1) + \Gamma_{h(Q)}(v, w_2)}{2} \right) = 0, \tag{1}$$

$$\sup_{v_1, v_2, w_2, w_3} \left(\Gamma_Q^2 \left(\frac{v_1 + v_2}{2}, \frac{w_1 + w_2}{2} \right) \dot{-} \left(\frac{\Gamma_Q(v_1, w_1) + \Gamma_Q(v_2, w_2)}{2} \right)^2 \right) = 0, \tag{2}$$

$$\sup_{v} \sup_{w} (|\Gamma_Q(v, w) - \Gamma_Q(iv, iw)| = 0, \tag{3}$$

$$\sup_{v} \sup_{w} (|\Gamma_Q(v, w) - \Gamma_Q(-v, -w)| = 0, \tag{4}$$

$$\sup_{w} \max\{\inf_{v} \Gamma_Q(v, w + iv), \inf_{v} \Gamma_Q(v, w - iv)\} = 0. \tag{5}$$

$$\sup_{v} \inf_{w} \Gamma_Q(v, w) = 0 \tag{6}$$

$$\inf_{v_1 v_2 \cdots v_{n_\lambda}} \sup_{w} \max \left(|\langle v_i | v_j \rangle - \delta_{ij}|, \Gamma_Q(v_i, \lambda v_i), \left\| \chi_{\{\lambda\}}(Q) w - \sum_{k=1}^{n_\lambda} \langle w \mid v_i \rangle v_i \right\| \right) = 0 \tag{7}$$

$$\inf_{v_1 v_2 \cdots v_n} \inf_{w_1, w_2 \cdots w_n} \max_{i,j=1,\cdots,n} \left(|\langle w_i \mid w_j \rangle - \delta_{ij}|, \|g(Q)v_i - w_i\| \right) = 0 \tag{8}$$

$$\sup_{v} \sup_{w} ((c_\nu \|v\| \dot{-} \|w\|) \dot{-} f(\Gamma_Q(v, \nu v + w))) = 0 \tag{9}$$

Condition (1) expresses the fact that $h(Q)$ is a function (cf. Lemma 6.2); conditions (2), (3), and (4) express the fact that Q is linear (cf. Lemma 6.3); condition 5 expresses that Q is essentially self-adjoint (cf. Fact 2.14); condition (6) expresses that $D(Q)$ is dense.

Theorem 6.11 *The class $\mathcal{K}_{(H, \Gamma_Q)}$ is exactly the class of all models of* $\mathsf{IHS}_{\sigma(Q)}$.

P r o o f. All continuous first order axioms guarantee that all models of $\mathsf{IHS}_{\sigma(Q)}$ are spectrally equivalent to (H, Γ_Q). Condition 5 says that, in each model $(H', \Gamma_{Q'})$ of $\mathsf{IHS}_{\sigma(Q)}$, the operator Q' is essentially self-adjoint. The Condition $\Gamma_{Q'} = 0$ implies that the graph of Q' is closed, so Q is a closed operator. Condition (6) implies says that in each model of $\mathsf{IHS}_{\sigma(Q)}$ the domain of the closed unbounded operator is dense in the Hilbert space. So, all the models of $\mathsf{IHS}_{\sigma(Q)}$ belong to $\mathcal{K}_{(H, \Gamma_Q)}$. By spectral theory, the converse is true, so both classes are the same. □

The theory $\mathsf{IHS}_{\sigma(Q)}$ is not a theory in continuous first order logic but in continuous $\mathcal{L}_{\omega_1 \omega}$ logic.

Now, we provide an example of a class $\mathcal{K}_{(H, \Gamma_Q)}$ that only has one model which clarifies why, in general $\mathcal{K}_{(H, \Gamma_Q)}$ is not the same as the class of models of $\mathrm{Th}(H, \Gamma_Q)$, the first order theory of (H, Γ_Q). This example is very similar to the quantum harmonic oscillator:

Example 6.12 Let $(H_{\mathbb{N}}, \Gamma_{Q_N})$ be a separable Hilbert space structure such that $\sigma(Q_N) = \sigma_{\mathrm{d}}(Q) = \mathbb{N}$ and for every $n \in \mathbb{N}$ the eigenspace corresponding to n has dimension 1.

Claim 6.13 *Let $\mathcal{U}$ a non principal ultrafilter over $\mathbb{N}$. Then $D(\prod_{\mathcal{U}} Q_{\mathbb{N}})$ is not dense in $\prod_{\mathcal{U}} H_{\mathbb{N}}$.*

P r o o f. Let $(v_n)_{n \in \mathbb{N}}$ be a sequence of vectors in $H_{\mathbb{N}}$ such that $\|v_n\| = 1$, $v_n \in D(Q_{\mathbb{N}})$ and $Qv_n = nv_n$ for every $n \in \mathbb{N}$. Then $(v_n)/\mathcal{U} \notin D(\prod_{\mathcal{U}} Q_{\mathbb{N}})$. Let $(w_n)/\mathcal{U}$ be such that $\|(v_n)/\mathcal{U} - (w_n)/\mathcal{U}\| < \varepsilon$, for some $\varepsilon > 0$. Then $\lim_{\mathcal{U}} \|w_n - v_n\| < \varepsilon$. That is, for some $B \in \mathcal{U}$ and for every $n \in B$, $\|w_n - v_n\| < \varepsilon$. Suppose, in addition, that for every $n \in B$, $w_n \in D(Q_{\mathbb{N}})$. Let $w_n = \sum_{k \geq 0} w_n^k$, where $w_n^k \in D(Q_{\mathbb{N}})$ and $Qw_n^k = kw_n^k$ for $k \in \mathbb{N}$. Then,

$0 \leq \sum_{k\in\mathbb{N},k\neq n} \|w_n^k\|^2 < \varepsilon^2$, and $\|w_n^n - v_n\| < \varepsilon$. If $\varepsilon < \frac{\|v_n\|}{2}$, then $\|Q(w_n^n)\| = \|Q(w_n^n - v_n + v_n)\| = \|Q(v_n - (v_n - w_n^n))\| = \|Q(v_n) - Q(v_n - w_n^n)\| \geq |\|Q(v_n) - Q(v_n - w_n^n)\|| = \left|n\|v_n\| - \frac{n}{2}\|v_n\|\right| = \frac{n}{2}$. So, $\|Q(w_n)\| \geq \frac{n}{2}$.

This means that $\lim_{\mathcal{U}} \|Q(w_n)\| = \infty$. So, $(w_n)/\mathcal{U} \notin D(\prod_{\mathcal{U}} Q_{\mathbb{N}})$. This way, we have proven that for some $(v_n)/\mathcal{U} \in \prod_{\mathcal{U}} H_{\mathbb{N}} \backslash D(\prod_{\mathcal{U}} Q_{\mathbb{N}})$, there exists an $\varepsilon > 0$ such that for every $(w_n)/\mathcal{U} \in \prod_{\mathcal{U}} H_{\mathbb{N}}$ such that $\|(v_n)/\mathcal{U} - (w_n)/\mathcal{U}\| < \varepsilon$, and $(w_n)/\mathcal{U} \notin D(\prod_{\mathcal{U}} Q_{\mathbb{N}})$. This proves that $\prod_{\mathcal{U}} Q_{\mathbb{N}}$ is not dense in $\prod_{\mathcal{U}} H_{\mathbb{N}}$. □

Claim 6.14 *The tuple $(\prod_{\mathcal{U}} H_{\mathbb{N}}, \Gamma_{\prod_{\mathcal{U}} Q_{\mathbb{N}}})$ does not belong to $\mathcal{K}_{(H_{\mathbb{N}},\Gamma_{Q_{\mathbb{N}}})}$.*

Proof. Claim 6.13 shows that $D(\prod_{\mathcal{U}} Q_{\mathbb{N}})$ is not dense in $\prod_{\mathcal{U}} H_{\mathbb{N}}$, and then $(\prod_{\mathcal{U}} H_{\mathbb{N}}, \Gamma_{\prod_{\mathcal{U}} Q_{\mathbb{N}}})$ does not belong to $\mathcal{K}_{(H_{\mathbb{N}},\Gamma_{Q_{\mathbb{N}}})}$. □

Claim 6.15 *The class $\mathcal{K}_{(H,\Gamma_Q)}$ is not, in general, first order axiomatizable.*

Proof. The previous theorem shows that $\mathcal{K}_{(H_{\mathbb{N}},\Gamma_{Q_{\mathbb{N}}})}$ is not closed under ultrapowers and, therefore, cannot be first order axiomatizable. □

7 Stability

In this section, we prove that the MAEC $\mathcal{K}_{(H,\Gamma_Q)}$ is superstable by counting types over sets and show that it is $\aleph_0$-stable up to perturbations. These are the statements of Theorems 7.7 and 7.9, respectively.

Theorem 7.1 *Let $v, w \in \tilde{H}$. Then $\tilde{H}_v$ is isometrically isomorphic to a Hilbert subspace of $\tilde{H}_w$ if and only if $\mu_v \ll \mu_w$.*

Proof. By the Radon-Nikodym Theorem, if $\mu_u \ll \mu_v$ then $\tilde{H}_v$ is isometrically equivalent to a Hilbert subspace of $\tilde{H}_w$. For the converse, if $\tilde{H}_v$ is isometrically equivalent to a Hilbert subspace of $\tilde{H}_w$, then v can be represented in $L^2(\mathbb{R}, \mu_w)$ by some function, and therefore, $\mu_u \ll \mu_v$. □

Recall that if $G \subseteq \tilde{H}$ is small, $S(G)$ denotes the set of Galois types in one variable over G.

Theorem 7.2 *Let $p, q \in S(\varnothing)$ and let $v, w \in \tilde{\mathcal{H}}$ such that $v \models p$ and $w \models q$, and $\mu_v \ll \mu_w$. Then, $d(p, q) = \|\mu_w - \mu_v\|$.*

Proof. If $\mu_u \ll \mu_v$, by Theorem 7.1, there exist $v' \models \mathrm{gatp}(v/\varnothing)$ such that $\tilde{H}_{v'} \leq \tilde{H}_w$ and there exists $f \in L^1(\sigma(Q), \mu_w)$ such that $d\mu_v = f d\mu_w$. Then $d|\mu_w - \mu_v| = |1 - f| d\mu_w$ and therefore $d(p, q) \leq \|\mu_w - \mu_v\|$. Since the $d(p, q)$ is the minimum of the distance between realizations of p and q, and this minimum occurs when $\tilde{H}_{v'} \leq \tilde{H}_w$, we have that $d(p, q) = \|\mu_w - \mu_v\|$. □

Theorem 7.3 *Let $p, q \in S(\varnothing)$ and let $v, w \in \tilde{H}$ be such that $v \models p$ and $w \models q$, and $\mu_v \perp \mu_w$. Then, $d(p, q) = \sqrt{\|\mu_v\|^2 + \|\mu_w\|^2}$.*

Proof. If $\mu_v \perp \mu_w$, by Theorem 7.1, no Hilbert subspace of $\tilde{H}_v$ is isometrically isomorphic to a Hilbert subspace of $\tilde{H}_w$. Then we can assume $\tilde{H}_v \perp \tilde{H}_w$ and therefore, $d(p, q) = \|v - w\| = \sqrt{\|v\|^2 + \|w\|^2} = \sqrt{\|\mu_v\|^2 + \|\mu_w\|^2}$. □

Theorem 7.4 *Let $p, q \in S(\varnothing)$ and let $v, w \in \tilde{\mathcal{H}}$ be such that $v \models p$ and $w \models q$, and $\mu_w = \mu_w^{\|} + \mu_w^{\perp}$ according to the Lebesgue Decomposition Theorem. Then, $d(p, q) = \sqrt{\|\mu_v - \mu_w^{\|}\|^2 + \|\mu_w^{\perp}\|^2}$.*

Proof. By Theorems 7.2 and 7.3. □

Theorem 7.5 *Let $G \subseteq \tilde{H}$ be small, let $p, q \in S(G)$ and let $v, w \in \tilde{H}$ be such that $u \models p$ and $v \models q$. Then,*

$$d(p, q) = \sqrt{[P_G(v) - P_G(w)]^2 + d^2(\mathrm{gatp}(P_G^{\perp} v/\varnothing), \mathrm{gatp}(P_G^{\perp} w/\varnothing))}$$

Proof. By Theorem 3.13. □

Corollary 7.6 *Let $G \subseteq \tilde{H}$ then* $\mathrm{dens}[S_1(G)] \leq |G| \times 2^{\aleph_0}$.

Proof. Clear from Theorems 3.13 and 7.5. □

Theorem 7.7 *$\mathcal{K}_{(H,\Gamma_Q)}$ is κ-stable for $\kappa \geq |\sigma|$.*

Proof. Clear from Corollary 7.6. □

Definition 7.8 A MAEC with a system or perturbations $(\mathcal{K}, \prec_{\mathcal{K}}, (\mathbb{F}_\varepsilon)_{\varepsilon \geq 0})$ is said to be $\aleph_0$-stable up to the system of perturbations if for every separable structure $\mathcal{M} \in \mathcal{K}$ there is a separable $\mathcal{N} \succ_{\mathcal{K}} \mathcal{M}$, such that for every $\varepsilon > 0$ and for every separable structure $\mathcal{N}' \succ_{\mathcal{K}} \mathcal{M}$, there is an ε-perturbation $f : \mathcal{N}' \to \mathcal{N}$ such that $f \restriction \mathcal{M} = \mathrm{Id}_{\mathcal{M}}$.

Theorem 7.9 *The MAEC $\mathcal{K}_{(H,\Gamma_Q)}$ is $\aleph_0$-stable up to the system of perturbations.*

Proof. Let $(H_0, \Gamma_{Q_0}) \in \mathcal{K}_{(H,\Gamma_Q)}$ be separable. Let Λ be a countable dense subset of $\sigma_{\mathrm{e}}(Q)$. Let

$$(H_1, \Gamma_{Q_1}) := (H_0, \Gamma_{Q_0}) \oplus \bigoplus_{\lambda \in \Lambda} (L^2(R, \delta_\lambda), M_\lambda).$$

Let $(H_2, \Gamma_{Q_2}) \succ (H_0, \Gamma_{Q_0})$. Let (H_2', Γ_{Q_2}') be the orthogonal complement of (H_0, Γ_{Q_0}) in (H_2, Γ_{Q_2}). By Theorem 6.11 (H_1, Γ_{Q_1}) and (H_2', Γ_{Q_2}') are approximately uniformly equivalent and therefore there is an ε-perturbation relating (H_1, Γ_{Q_1}) and (H_2, Γ_{Q_2}). □

In the previous proof, recall that M_λ is the multiplication by λ.

8 Spectral independence

In this section, we define an independence relation in $\mathcal{K}_{(H,\Gamma_Q)}$, called *spectral independence*. Theorem 8.6 states that this relation has the same properties as non-forking for superstable first order theories, while Theorems 8.8 and 8.9 state that this relation characterize non-splitting.

Definition 8.1 Let $v_1, \dots, v_n \in \tilde{H}$ and let $F, G \subseteq \tilde{H}$. We say that $v_1, \dots, v_n$ are *spectrally independent* from G over F if for all $i \leq n\, P_{\mathrm{acl}(F)} v_i = P_{\mathrm{acl}(F \cup G)} v_i$ and denote it by $v_1, \dots, v_n \mathop{\smile\!\!\!\!\!\!\!\;|}\nolimits^{*}_{F} G$.

Remark 8.2 Let $v, w \in \tilde{H}$. Then v is independent from w over $\varnothing$ if and only if $\tilde{H}_{v_{\mathrm{e}}} \perp \tilde{H}_{w_{\mathrm{e}}}$ and denote it $v \mathop{\smile\!\!\!\!\!\!\!\;|}\nolimits^{*}_{\varnothing} w$.

Remark 8.3 Let $v, w \in \tilde{H}$. Let $G \subseteq \tilde{H}$ be small. Then v is independent from w over G if and only if $\tilde{H}_{P^{\perp}_{\mathrm{acl}(G)}(v)} \perp \tilde{H}_{P^{\perp}_{\mathrm{acl}(G)}(w)}$ and denote it $v \mathop{\smile\!\!\!\!\!\!\!\;|}\nolimits^{*}_{G} w$.

Remark 8.4 Let $\bar{v} \in H^n$ and $E, F \subseteq H$. Then $\bar{v} \mathop{\smile\!\!\!\!\!\!\!\;|}\nolimits^{*}_{E} F$ if and only if for every $j = 1, \dots, n v_j \mathop{\smile\!\!\!\!\!\!\!\;|}\nolimits^{*}_{E} F$ that is, for all $j = 1, \dots, n P_{\mathrm{acl}(E)}(v_j) = P_{\mathrm{acl}(E \cup F)}(v_j)$.

Theorem 8.5 *Let $F \subseteq G \subseteq H$, $p \in S_n(F) q \in S_n(G)$ and $\bar{v} = (v_1, \dots, v_n)$, $\bar{w} = (v_1, \dots, v_n) \in H^n$ be such that $p = \mathrm{gatp}(\bar{v}/F)$ and $q = \mathrm{gatp}(\bar{w}/G)$. Then q is an extension of p such that $\bar{w} \mathop{\smile\!\!\!\!\!\!\!\;|}\nolimits^{*}_{F} G$ if and only if the following conditions hold:*

1. *For every $j = 1, \dots, n$, $P_{\mathrm{acl}(F)}(v_j) = P_{\mathrm{acl}(G)}(w_j)$ and*
2. *for every $j = 1, \dots, n$, $\mu_{P^{\perp}_{\mathrm{acl}(F)} v_j} = \mu_{P^{\perp}_{\mathrm{acl}(G)} w_j}$.*

Proof. Clear from Theorem 3.13 and Remark 8.3 □

Theorem 8.6 *The relation $\mathop{\smile\!\!\!\!\!\!\!\;|}\nolimits^{*}$ satisfies local character, finite character, transitivity of independence, symmetry, existence, and stationarity.*

Proof. By Remark 8.4, to prove local character, finite character and transitivity it is enough to show them for the case of a 1-tuple.

Local character. Let $v \in H$ and $G \subseteq \tilde{H}$. Let $w = (P_{\mathrm{acl}(G)}(v))_{\mathrm{e}}$. Then there exist a sequence of $(l_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$, a sequence $(f_1^k, \dots, f_{l_k}^k)_{k \in \mathbb{N}}$ of finite tuples of bounded Borel funtions of $\mathbb{R}$ and a sequence of finite tuples $(e_1^k, \dots, e_{l_k}^k)_{k \in \mathbb{N}} \subseteq G$ such that if $w_k := \sum_{j=1}^{l_k} f_j^k(\tilde{Q}) e_j^k$ for $k \in \mathbb{N}$, then $w_k \to w$ when $k \to \infty$. Let $E_0 = \{e_j^k \mid j = 1, \dots, l_k$ and $k \in \mathbb{N}\}$. Then $v \mathop{\smile\!\!\!\!\!\!\!\;|}\nolimits^{*}_{E_0} E$ and $|E_0| = \aleph_0$.

Finite character. We show that for $v \in H$, $E, F \subseteq \tilde{H}$, $v \perp^*_E F$ if and only if $v \perp^*_E F_0$ for every finite $F_0 \subseteq F$. The left to right direction is clear. For right to left, suppose that $v \not\perp^*_E F$. Let $w = P_{\mathrm{acl}(E \cup F)}(v) - P_{\mathrm{acl}(E)}(v)$. Then $w \in \mathrm{acl}(E \cup F) \backslash \mathrm{acl}(E)$.

As in the proof of local character, there exist a sequence of pairs $(l_k, n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}^2$, a sequence $(g^k_1, \dots, g^k_{l_k+n_k})_{k \in \mathbb{N}}$ of finite tuples of bounded Borel functions on $\mathbb{R}$, and a sequence of finite tuples $(e^k_1, \dots, e^k_{l_k}, f^k_1, \dots, f^k_{n_k})_{k \in \mathbb{N}}$ such that $(e^k_1, \dots, e^k_{l_k}) \subseteq E$, $(f^k_1, \dots, f^k_{n_k})_{k \in \mathbb{N}} \subseteq F$ and if $w_k := \sum_{j=1}^{l_k} g^k_j(\tilde{Q}) e^k_i + \sum_{j=1}^{n_k} g^k_{l_k+j}(\tilde{Q}) f^k_j$ for $k \in \mathbb{N}$, then $w_k \to w$ when $k \to \infty$.

If $v \not\perp^*_E F$, then $w = P_{\mathrm{acl}(E \cup F)}(v) - P_{\mathrm{acl}(E)}(v) \neq 0$. For $\varepsilon = \|w\| > 0$ there is k_ε such that if $k \geq k_\varepsilon$ then $\|w - w_k\| < \varepsilon$. Let $F_0 := \{f^1_1, \dots, f^{n_{k_\varepsilon}}_{k_\varepsilon}\}$ Then F_0 is a finite subset such that $v \not\perp^*_E F_0$.

Transitivity of independence. Let $v \in H$ and $E \subseteq F \subseteq G \subseteq H$. If $v \perp^*_E G$ then $P_{\mathrm{acl}(E)}(v) = P_{\mathrm{acl}(G)}(v)$. It is clear that $P_{\mathrm{acl}(E)}(v) = P_{\mathrm{acl}(F)}(v) = P_{\mathrm{acl}(G)}(v)$ so $v \perp^*_E F$ and $v \perp^*_F G$. Conversely, if $v \perp^*_E F$ and $v \perp^*_F G$, we have that $P_{\mathrm{acl}(E)}(v) = P_{\mathrm{acl}(F)}(v)$ and $P_{\mathrm{acl}(F)}(v) = P_{\mathrm{acl}(G)}(v)$. Then $P_{\mathrm{acl}(E)}(v) = P_{\mathrm{acl}(G)}(v)$ and $v \perp^*_E G$.

Symmetry. Symmetry is clear from Remark 8.3.

Invariance. Let U be an automorphism of $(\tilde{H}, \Gamma_{\tilde{Q}})$. Let $\bar{v} = (v_1, \dots, v_n)$,$\bar{w} = (w_1, \dots, w_n) \in \tilde{H}^n$ and $G \subseteq \tilde{H}$ be such that $\bar{v} \perp^*_G \bar{w}$. By Remark 8.3, this means that for every $j, k = 1, \dots, n \tilde{H}_{P^\perp_{\mathrm{acl}(G)}(v_j)} \perp \tilde{H}_{P^\perp_{\mathrm{acl}(G)}(w_k)}$. It follows that for every $j, k = 1, \dots, n \tilde{H}_{P^\perp_{\mathrm{acl}(UG)}(Uv_j)} \perp \tilde{H}_{P^\perp_{\mathrm{acl}(UG)}(Uw_k)}$ and, again by Remark 8.3, $Uv \perp^*_{\mathrm{acl}(UG)} Uw$.

Existence. Let $F \subseteq G \subseteq \tilde{H}$ be small sets. We show, by induction on n, that for every $p \in S_n(F)$, there exists $q \in S_n(G)$ such that q is an $\perp^*$-independent extension of p.

Case $n = 1$. Let $v \in \tilde{H}$ be such that $p = \mathrm{gatp}(v/F)$ and let $(H', \Gamma_{Q'}) \in \mathcal{K}_{(H, \Gamma_Q)}$ be a structure containing v and G. Define

$$H'' := H' \oplus L^2(\mathbb{R}, \mu_{(P^\perp_{\mathrm{acl}(F)} v)_{\mathrm{e}}}),$$

$$Q'' := Q' \oplus M_{f_{(P^\perp_{\mathrm{acl}(F)} v)_{\mathrm{e}}}}$$

and

$$v' := P_{\mathrm{acl}(F)} v + (1)_{\sim_{\mu_{(P^\perp_{\mathrm{acl}(F)} v)_{\mathrm{e}}}}}$$

Then $(H'', \Gamma_{Q''}) \in \mathcal{K}_{(H, \Gamma_Q)}$, $v' \in H''$ and, by Theorem 8.5, the type $\mathrm{gatp}(v'/G)$ is a $\perp^*$-independent extension of $\mathrm{gatp}(v/F)$.

Induction step. Now, let $\bar{v} = (v_1, \dots, v_n, v_{n+1}) \in \tilde{H}^{n+1}$. By induction hypothesis, there are $v'_1, \dots, v'_n \in H$ such that $\mathrm{gatp}(v'_1, \dots, v'_n/G)$ is a $\perp^*$-independent extension of $\mathrm{gatp}(v_1, \dots, v_n/F)$. Let U be a monster model automorphism fixing F pointwise such that for every $j = 1, \dots, n$, $U(v_j) = v'_j$. Let $v'_{n+1} \in \tilde{H}$ be such that $\mathrm{gatp}(v'_{n+1}/G v'_1 \cdots v'_n)$ is a $\perp^*$-independent extension of $\mathrm{gatp}(U(v_{n+1})/F v'_1, \cdots v'_n)$. Then, by transitivity, $\mathrm{gatp}(v'_1, \dots, v'_n, v'_{n+1}/G)$ is a $\perp^*$-independent extension of $\mathrm{gatp}(v_1, \dots, v_n, v_{n+1}/F)$.

Stationarity. Let $F \subseteq G \subseteq \tilde{H}$ be small sets. We show, by induction on n, that for every $p \in S_n(F)$, if $q \in S_n(G)$ is a $\perp^*$-independent extension of p to G then $q = p'$, where p' is the $\perp^*$-independent extension of p to G built in the proof of existence.

Case $n = 1$. Let $v \in H$ be such that $p = \mathrm{gatp}(v/F)$, and let $q \in S(G)$ and $w \in H$ be such that $w \models q$. Let v' be as in previous item. Then, by Theorem 8.5 we have that:

1. $P_{\mathrm{acl}(F)} v = P_{\mathrm{acl}(G)} v' = P_{\mathrm{acl}(G)} w$
2. $\mu_{P^\perp_{\mathrm{acl}(F)} v} = \mu_{P^\perp_{\mathrm{acl}(G)} w} = \mu_{P^\perp_{\mathrm{acl}(G)} v'}$

This means that $P_{\mathrm{acl}(G)} v' = P_{\mathrm{acl}(G)} w$, $\mu_{P^\perp_{\mathrm{acl}(G)} w} = \mu_{P^\perp_{\mathrm{acl}(G)} v'}$ and, therefore $q = tp(v'/G) = p'$.

Induction step. Let $\bar{v} = (v_1, \dots, v_n, v_{n+1})$, $\bar{v}' = (v'_1, \dots, v'_n, v'_{n+1})$ and $\bar{w} = (w_1, \dots, w_n) \in \tilde{H}$ be such that $\bar{v} \models p$, $\bar{v}' \models p'$ and $\bar{w} \models q$. By transitivity, we have that $\mathrm{gatp}(v'_1, \dots, v'_n/G)$ and $\mathrm{gatp}(w_1, \dots, w_n/G)$ are $\perp^*$-independent extensions of $\mathrm{gatp}(v_1, \dots, v_n/F)$. By induction hypothesis, $\mathrm{gatp}(v'_1, \dots, v'_n/G) = \mathrm{gatp}(w_1, \dots, w_n/G)$. Let U be a monster model automorphism fixing F pointwise such that for every

$j = 1, \ldots, n$, $U(v_j) = v'_j$ and let U' a monster model automorphism fixing G pointwise such that for every $j = 1, \ldots, n$, $U'(v_j) = w_j$. Again by transitivity,

$$\operatorname{gatp}(U(w_{n+1})/Gv_1 \cdots v_n)$$

and

$$\operatorname{gatp}(v'_{n+1}/Gv_1, \cdots v_n)$$

are $\downarrow^*$-independent extensions of $\operatorname{gatp}(v_{n+1}/Gv_1, \cdots v_n)$.

By the case $n = 1$,

$$\operatorname{gatp}(U^{-1}(v'_{n+1})/U^{-1}Gv_1 \cdots v_n) = \operatorname{gatp}((U' \circ U)^{-1}(w_{n+1})/U^{-1}Gv_1, \cdots v_n)$$

and therefore

$$p' = \operatorname{gatp}(v'_1, \ldots, v'_n v'_{n+1}/G) = \operatorname{gatp}(w_1, \ldots, w_n, w_{n+1}/G) = q.$$

□

Definition 8.7 Let $\mathcal{K}$ be an homogeneous MAEC with monster model $\mathcal{M}$. Let $B \subseteq A \subseteq M$ and let $a \in M$. The type $\operatorname{gatp}(a/A)$ is said to split over B if there are $b, c \in A$ such that

$$\operatorname{gatp}(b/B) = \operatorname{gatp}(c/B)$$

but

$$\operatorname{gatp}(b/Ba) \neq \operatorname{gatp}(c/Ba)$$

Theorem 8.8 *Let $v \in \tilde{H}$ and let $F \subseteq G \subseteq \tilde{H}$. If $\operatorname{gatp}(v/G)$ splits over F then $v \not\downarrow^*_F G$.*

Proof. If $\operatorname{gatp}(v/G)$ splits over F, then there are two vectors w_1 and $w_2 \in G$ such that $\operatorname{gatp}(w_1/F) = \operatorname{gatp}(w_2/F)$ but $\operatorname{gatp}(w_1/Fv) \neq \operatorname{gatp}(w_2/Fv)$. Then, either $\operatorname{gatp}(P^{\perp}_{\operatorname{acl}(Fv)} w_1/\varnothing) \neq \operatorname{gatp}(P^{\perp}_{\operatorname{acl}(Fv)} w_2/\varnothing)$ or $P_{\operatorname{acl}(Fv)} w_1 \neq P_{\operatorname{acl}(Fv)} w_2$. Let us consider each case:

Case 1: $\operatorname{gatp}(P^{\perp}_{\operatorname{acl}(Fv)} w_1/\varnothing) \neq \operatorname{gatp}(P^{\perp}_{\operatorname{acl}(Fv)} w_2/\varnothing)$. Since

$$P^{\perp}_{\operatorname{acl}(Fv)} w_1 = P^{\perp}_{\operatorname{acl}(F)} w_1 - P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_1$$

and

$$P^{\perp}_{\operatorname{acl}(Fv)} w_2 = P^{\perp}_{\operatorname{acl}(F)} w_2 - P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_2,$$

this means that

$$\operatorname{gatp}(P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_1/\varnothing) \neq \operatorname{gatp}(P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_2/\varnothing)$$

So, either $P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_1 \neq 0$ or $P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_2 \neq 0$. Let us suppose without loss of generality that $P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_1 \neq 0$. Then $P_{w_1}(P^{\perp}_{\operatorname{acl}(F)} v_e) \neq 0$, which implies that $P_{\operatorname{acl}(F)} v \neq P_{\operatorname{acl}(Fw_1)} v$. That is, $v \not\downarrow^*_F w_1$ and by transitivity, $v \not\downarrow^*_F G$.

Case 2: $P_{\operatorname{acl}(Fv)} w_1 \neq P_{\operatorname{acl}(Fv)} w_2$. Since

$$P_{\operatorname{acl}(Fv)} w_1 = P_{\operatorname{acl}(F)} w_1 + P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_1$$

and

$$P_{\operatorname{acl}(Fv)} w_2 = P_{\operatorname{acl}(F)} w_2 + P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_2,$$

this means that $P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_1 \neq P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_2$ and, therefore either $P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_1 \neq 0$ or $P_{P^{\perp}_{\operatorname{acl}(F)} v_e} w_2 \neq 0$. As in previous item, this implies that $v \not\downarrow^*_F G$. □

Theorem 8.9 *Let $v \in \tilde{H}$ and $F \subseteq G \subseteq \tilde{H}$ such that $F = \operatorname{acl}(F)$ and G is $|F|$-saturated. If $v \not\downarrow^*_F G$, then $\operatorname{gatp}(v/G)$ splits over F.*

Proof. If $v \not\downarrow^*_F G$ then $w := P_G v - P_F v \neq 0$ and $w \perp F$. Since G is $|F|$-saturated, there is $w' \in G$ such that $\operatorname{gatp}(w/F) = \operatorname{gatp}(w'/F)$ and $w' \perp P_G v$. Since $\langle v \mid w \rangle \neq 0$, $P_v w \neq 0$, while $P_v w' = 0$. □

Definition 8.10 Let $\varepsilon > 0$, $v \in \tilde{H}$ and let $F, G \subseteq \tilde{H}$. We say that v is ε-spectrally independent from G over F if $\|P_{\mathrm{acl}(F\cup G)}v - P_{\mathrm{acl}(F)}v\| \leq \varepsilon$ and denote it $v \mathop{\smile\!\!\!\!\!\mid}^{\varepsilon}_{F} G$.

Theorem 8.11 *The relation* $\mathop{\smile\!\!\!\!\!\mid}^{\varepsilon}$ *satisfies the following properties:*
Local character: *Let $v \in H$, $G \subseteq \tilde{H}$ and $\varepsilon > 0$. Then there is a finite $G_0 \subseteq G$ such that $v \mathop{\smile\!\!\!\!\!\mid}^{\varepsilon}_{G_0} G$.*
Monotonicity of independence: *Let $v \in H$ and $D \subseteq E \subseteq F \subseteq G \subseteq H$. If $v \mathop{\smile\!\!\!\!\!\mid}^{\varepsilon}_{D} G$ then $v \mathop{\smile\!\!\!\!\!\mid}^{\varepsilon}_{E} F$*

Proof. *Local character.* Let $v \in H$, $G \subseteq \tilde{H}$ and $\varepsilon > 0$. Let w, $(l_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$, $(e^k_1, \ldots, e^k_{l_k})_{k\in\mathbb{N}} \subseteq G$, $(f^k_1, \ldots, f^k_{l_k})_{k\in\mathbb{N}}$ and w_k for $k \in \mathbb{N}$ be as in the proof of local character of $\mathop{\smile\!\!\!\!\!\mid}^{*}$ in Theorem 8.6. Since $w_k \to w$ when $k \to \infty$, there is a $k_1 \in \mathbb{Z}$ such that $\|w_k - w\| < \varepsilon$ for all $k \geq k_1$. Let $G_o := \{e^k_j \mid j = 1, \ldots, l_k \text{ and } k \leq k_1\}$. Then, $v \mathop{\smile\!\!\!\!\!\mid}^{*}_{G_0} G$.

Monotonicity of independence. Let $v \in H$ and $D \subseteq E \subseteq F \subseteq G \subseteq H$ and $\varepsilon > 0$. If $v \mathop{\smile\!\!\!\!\!\mid}^{\varepsilon}_{D} G$ then $\varepsilon \geq \|P_{\mathrm{acl}(D\cup G)}v - P_{\mathrm{acl}(D)}v\| = \|P_{\mathrm{acl}(G)}v - P_{\mathrm{acl}(D)}v\| \geq \|P_{\mathrm{acl}(f)}v - P_{\mathrm{acl}(E)}v\|$. Therefore $v \mathop{\smile\!\!\!\!\!\mid}^{\varepsilon}_{E} F$. □

Theorem 8.11 shows that the class $\mathcal{K}_{(H,\Gamma_Q)}$ is superstable.

Definition 8.12 Let $\bar{v} = (v_1, \ldots, v_n) \in H^n$ and $G \subseteq H$. A *canonical base* for the type $\mathrm{gatp}(\bar{v}/G)$ is a set $F \subseteq H_G$ which is fixed pointwise by the parallelism class of Morley sequences in $\mathrm{gatp}(\bar{v}/G)$ and such that $\bar{v} \mathop{\smile\!\!\!\!\!\mid}^{*}_{F} G$.

Theorem 8.13 *Let $\bar{v} = (v_1, \ldots, v_n) \in H^n$ and $G \subseteq H$. Then* $\mathrm{Cb}(\mathrm{gatp}(\bar{v}/G)) := \{(P_G v_1, \ldots, P_G v_n)\}$ *is a canonical base for the type* $\mathrm{gatp}(\bar{v}/G)$.

Proof. First of all, we consider the case of a 1-tuple. By Theorem 8.5 $\mathrm{gatp}(v/G)$ does not fork over $\mathrm{Cb}(\mathrm{gatp}(v/G))$. Let $(v_k)_{k<\omega}$ a Morley sequence for $\mathrm{gatp}(v/G)$. We have to show that $P_G v \in \mathrm{dcl}((v_k)_{k<\omega})$. By Theorem 8.5, for every $k < \omega$ there is a vector w_k such that $v_k = P_G v + w_k$ and $w_k \perp \mathrm{acl}(\{P_G v\} \cup \{w_j \mid j < k\})$. This means that for every $k < \omega$, $w_k \in H_{\mathrm{e}}$ and for all $j, k < \omega$, $H_{w_j} \perp H_{w_k}$. For $k < \omega$, let $v'_k := \frac{v_1+\cdots+v_k}{n} = P_G v + \frac{w_1+\cdots+w_k}{n}$. Then for every $k < \omega$, $v'_k \in \mathrm{dcl}((v_k)_{k<\omega})$. Since $v'_k \to P_{\mathrm{e}} v$ when $k \to \infty$, we have that $P_G v \in \mathrm{dcl}((v_k)_{k<\omega})$.

For the case of a general n-tuple, by Remark 8.4, it is enough to repeat previous argument in every component of $\bar{v}$. □

9 Orthogonality and domination

In this section, we characterize domination, orthogonality of types in terms of absolute continuity and mutual singularity between spectral measures.

Theorem 9.1 *Let $p, q \in S_1(\varnothing)$, let $v \models p$ and $w \models q$. Then, $p \perp^a q$ if and only if $\mu_{v_{\mathrm{e}}} \perp \mu_{w_{\mathrm{e}}}$.*

Proof. $p \perp^a q$ if and only if $\tilde{H}_{v'_e} \perp \tilde{H}_{w'_e}$ for all $v'_e \models p$ and $w'_e \models q$. By Lesbesgue decomposition theorem $\mu_{w_{\mathrm{e}}} = \mu^{\parallel}_{v_{\mathrm{e}}} + \mu^{\perp}_{v_{\mathrm{e}}}$ where, $\mu^{\parallel}_{v_{\mathrm{e}}} \ll \mu_{v_{\mathrm{e}}}$ and $\mu^{\perp}_{v_{\mathrm{e}}} \perp \mu_{v_{\mathrm{e}}}$. $\mu^{\parallel}_{v_{\mathrm{e}}} \neq 0$ if and only if there is a choice of $v' \models p$ and $w' \models q$ such that $\tilde{H}_{v'_{\mathrm{e}}} \cap \tilde{H}_{w'_{\mathrm{e}}} \neq \{0\}$ and therefore $\tilde{H}_{v'_{\mathrm{e}}} \not\perp \tilde{H}_{w'_{\mathrm{e}}}$. □

Corollary 9.2 *Let $G \subseteq \tilde{H}$ be small. Let $p, q \in S_1(G)$, let $v \models p$ and $w \models q$. Then, $p \perp^a_G q$ if and only if $\mu_{P^{\perp}_G v_{\mathrm{e}}} \perp \mu_{P^{\perp}_G w_{\mathrm{e}}}$*

Proof. Clear from Theorem 9.1. □

Corollary 9.3 *Let $G \subseteq H$ be small. Let $p, q \in S_1(G)$. Then, $p \perp^a q$ if and only if $p \perp q$.*

Proof. Clear from Corollary 9.2. □

Theorem 9.4 *Let $p, q \in S_1(\varnothing)$, let $v \models p$ and $w \models q$. Then, $p \rhd_{\varnothing} q$ if and only if $\mu_{v_{\mathrm{e}}} \gg \mu_{w_{\mathrm{e}}}$.*

Proof. Suppose $p \rhd_{\varnothing} q$. Suppose that v and w are such that if $v \mathop{\smile\!\!\!\!\!\mid}^{*}_{\varnothing} G$ then $w \mathop{\smile\!\!\!\!\!\mid}^{*}_{\varnothing} G$ for every $G \subseteq \tilde{H}$. Then for every G if $\tilde{H}_{v_{\mathrm{e}}} \perp \tilde{H}_G$ then $\tilde{H}_{w_{\mathrm{e}}} \perp \tilde{H}_G$. This means $\tilde{H}_{w_{\mathrm{e}}} \subseteq \tilde{H}_{v_{\mathrm{e}}}$ and $\tilde{H}_{w_{\mathrm{e}}}$ is unitarily equivalent to some Hilbert subspace of $\tilde{H}_{w_{\mathrm{e}}}$ and by Theorem 7.1 $\mu_{w_{\mathrm{e}}} \ll \mu_{v_{\mathrm{e}}}$. □

Corollary 9.5 *Let E, F, and G be small subsets of $\tilde{H}$ and $p \in S_1(F)$ and $q \in S_1(G)$ two stationary types. Then $p \rhd_E q$ if and only if there exist $vw \in \tilde{\mathcal{H}}$ such that* $\mathrm{gatp}(v/E)$ *is a non-forking extension of p,* $\mathrm{gatp}(w/E)$ *is a non-forking extension of q and $\mu_{P^{\perp}_{\mathrm{acl}(F)} v} \gg \mu_{P^{\perp}_{\mathrm{acl}(F)} w}$.*

P r o o f . Clear from Theorem 9.4. □

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